Periodic orbits and Arnold diffusion

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We compute the Arnold diffusion time in initially hyperbolic Hamiltonian systems H_μ, where $0 < \mu << 1$ is the perturbing parameter. We prove that in the Arnold example this time is of order $\mu \log\left(\frac{1}{\mu}\right)$. Our method is based on the construction of a dual chain of hyperbolic periodic orbits surrounding a given transition chain of partially hyperbolic tori, whose parameters (angles, periods) can be related to parameters (diophantine condition, angles) of the original chain of tori. Using Easton's method of windows, we give a general formula for the time of drift along such a chain. We then deduce the result for chain of partially hyperbolic tori.

Key Words: Hamiltonian systems, hyperbolic tori, Symbolic Dynamics.

1. INTRODUCTION

Let H_μ be an initially hyperbolic Hamiltonian system, where μ is a small parameter.

Let $\mathcal{T} = \{T_1, \ldots, T_N\}$ be a family of Graff tori for H_μ such that $W^+(T_i)$ and $W^-(T_{i+1})$ intersect transversally in a given energy manifold $\mathcal{H}$. As Graff tori satisfy the obstruction property ([20],[8]), $\mathcal{T}$ is a transition chain ([2]). Let U_1 (resp. U_N) be an arbitrary small neighbourhood of $W^+(T_1)$ (resp. $W^-(T_N)$). There exists an orbit $\xi(t)$ of H_μ, and a time $T > 0$, such that $\xi(0) \in U_1$ and $\xi(T) \in U_N$. Following [3], we call T the *Arnold diffusion time*.

Many attempts have been made to compute the Arnold diffusion time. We refer to [3] and [5] for variational methods, and to [20] and [9]. In this paper, we describe a new geometrical method based on chains of hyperbolic periodic orbits.

Let us assume that the stable and unstable manifold of each tori intersect transversally along an homoclinic orbit. Then, following [10], there exists a hyperbolic invariant set near T_i, on which the dynamics is conjugated to a shift on an infinite set of symbols. This result allows us to construct a dual chain of hyperbolic periodic orbits surrounding the given chain of tori (see [10]).

In this paper, we compute the time of drift along a general chain of hyperbolic orbits using Easton's method of windows [14]. Using [11], we give a quantitative construction of the dual chain of hyperbolic periodic orbits surrounding a given chain of partially hyperbolic tori for an initially hyperbolic Hamiltonian system. Using this chain, we prove that the Arnold diffusion time is of order $O\left(\frac{1}{\mu}\log\frac{1}{\mu}\right)$, which is the best estimate obtained for this problem (see however [5]).

2. TIME OF DRIFT ALONG A CHAIN OF HYPERBOLIC PERIODIC ORBITS

Assuming the existence of a chain of hyperbolic periodic orbits, and under some quantitative informations about there period, and the splitting of the invariant manifolds, we compute the time of instability. We use the Easton method of windows [14].

2.1. Notations and main result

In this paper, we shall denote by $\|\ .\ \|_1$ the algebra norm of matrix space $M_{d,d}(\mathbb{R})$ defined by $\| A \|_1 = \sum_{i=1}^{d}(\sum_{j=1}^{d} | a_{i,j} |)$, for $A \in M_{d,d}(\mathbb{R})$ and by $\|\ .\ \|_\infty$ the usual supremum norm of $\mathbb{R}^d$.

(μ,τ,σ)-family of hyperbolic periodic orbits. - For μ sufficiently small, there exists a family $O_1,\dots,O_{N(\mu)}$ of hyperbolic periodic orbits such that :

(a_1) The unstable manifold $W^-(O_i)$ of O_i and the stable manifold $W^+(O_{i+1})$ of O_{i+1}, intersect transversaly in a given energy surface $\mathcal{H}$.

Let S_i be a Poincaré section associated to the hyperbolic periodic orbit O_i, $i = 1, \ldots, N(\mu)$. We denote by f_i the first return map in S_i. Let $\tau(\mu) \in \mathbb{N}$ be the period of each orbit O_i. Then the map $f_i^\tau = F_i$ has a hyperbolic fixed point.

Remark 2. 1. In this paper, we choose $\tau_i(\mu) = \tau(\mu)$ for all $i = 1, \ldots, N(\mu)$. Of course, the same can be done with a non uniform period.

We can always find a coordinates system such that F_i takes the form

$$F_i(s,u) = F_i^l(s,u) + r_i(s,u),$$

where F_i^l is the linear part of F_i defined by

$$F_i^l(s,u) = (\Lambda_i(\mu)s, \Lambda_i^{-1}(\mu)u)$$

with $s \in \mathbb{R}^m$, $u \in \mathbb{R}^m$, $\Lambda_i \in M_{m,m}(\mathbb{R})$ is a diagonal matrix whose eigenvalues are $\lambda_1^i(\mu), \ldots, \lambda_m^i(\mu)$ with $0 < \lambda_j^i(\mu) < 1$ for all $\mu > 0$ and $j = 1, \ldots, m$ and r_i is of order 2 in s, u.

(a_2) We assume that $r_i(s,u) = O_2(su)$,

Remark 2. 2. Assumption (a_2) follows from J. Moser's theorem [21] for $m = 1$.

We denote:

$$\begin{aligned} \gamma(\mu) &= \max_{i=1,\ldots,N(\mu)} \parallel \Lambda_i \parallel_\infty \\ \rho^{-1}(\mu) &= \max_{i=1,\ldots,N(\mu)} \parallel \Lambda_i^{-1} \parallel_\infty \end{aligned} \tag{1}$$

We denote by Γ_i the heteroclinic orbit in $W^+(O_{i+1}) \cap W^-(O_i)$. We assume that Γ_i intersects the section S_i in $p_i^- \in W^-(O_i)$, $p_i^- = (0, u_i^-)$, and the section S_{i+1} in $p_{i+1}^+ \in W^+(O_{i+1})$, $p_{i+1}^+ = (s_{i+1}^+, 0)$. We define two neighbourhoods of p_i^- and p_i^+ as

$$\begin{aligned} B_i^+ &= \{(s,u) \in \mathbb{R}^m \times \mathbb{R}^m; \parallel s - s_i^+ \parallel_\infty < \mu, \parallel u \parallel_\infty < \mu\}, \\ B_i^- &= \{(s,u) \in \mathbb{R}^m \times \mathbb{R}^m; \parallel s \parallel_\infty < \mu, \parallel u - u_i^- \parallel_\infty < \mu\}. \end{aligned} \tag{2}$$

The *section map* $\Theta_i : B_{i+1}^+ \to B_i^-$ is defined by

$$\Theta_i(x) = \Theta_i^l(x) + O(h^2), \tag{3}$$

where Θ_i^l is the linear part of Θ_i defined by

$$\Theta_i^l(x) = p_i^- + \Pi_i h$$

with $h = x - p^+_{i+1}$,

(a_3) We assume that Π_i takes the form

$$\Pi_i(\mu) = \begin{pmatrix} \sigma_i & a_i \\ b_i & I \end{pmatrix} \quad (4)$$

where a_i, $b_i \in M_{m,m}(\mathbb{R})$ are diagonal matrix such that $det(a_i \times b_i) \neq 0$, I is the $m \times m$ identity matrix, and $\sigma^i \in M_{m,m}(\mathbb{R})$ is a diagonal *splitting* matrix which is a function of μ and

$$\sigma^i(\mu) \to 0 \ \ if \ \ \mu \to 0$$

.

Remark 2. 3. The form of $\Pi_i(\mu)$ can be understand for $m = 1$ by the following conditions :
(i) For $\sigma = 0$, we have $W^+(O) = W^-(O)$,
(ii) For $\sigma \neq 0$, we have a transversal intersection between $W^+(O)$ and $W^-(O)$, with a splitting of size σ.

Condition (i) implies that $\Pi_i(0)(1, 0) = (0, *)$, where $* \neq 0$. Hence, we must have a matrix $\Pi(0)$ of the form

$$\Pi_i(0) = \begin{pmatrix} 0 & a \\ b & I \end{pmatrix}, \quad (5)$$

where $det(a \times b) \neq 0$.
Condition (ii) implies the given form of $\Pi_i(\mu)$.

(a_4) We assume that there exists positive constants α, β, C^-, C^+, K independent of μ such that, for all $i = 1, \dots, N(\mu)$ we have

$$\begin{array}{l} \parallel a_i \parallel_\infty < \alpha \\ \parallel b_i \parallel_\infty < \beta \\ \parallel (a_i b_i)^{-1} \parallel_\infty < K \\ \parallel u_i^- \parallel_\infty \leq C^- \\ \parallel s_i^+ \parallel_\infty \leq C^+ \end{array} \quad (6)$$

(a_5) We assume that there exists positive constants σ and ν dependent of μ such that, for all $i = 1, \dots, N(\mu)$ we have

$$\begin{array}{l} \parallel \sigma_i(\mu) \parallel_\infty < \sigma(\mu) \\ \parallel \sigma_i^{-1}(\mu) \parallel_\infty < \nu^{-1}(\mu) \end{array}$$

In the following, we denote by $\mathcal{P} = (\alpha, \beta, C^-, C^+, K, \rho, \gamma)$ the parameters of a given (μ, τ, σ) chain of hyperbolic periodic orbits. Then we deduce the main result of this section :

THEOREM 2.1. *For all (μ, τ, σ)-family of chain of hyperbolic periodic orbits, with μ sufficiently small, there exists an orbit $\zeta(t)$ of H_μ such that $\zeta(0) \in B_1^+$ and $\zeta(T) \in B_{N(\mu)}^-$ with*

$$T = C(\mathcal{P})N(\mu)\tau(\mu)\frac{\log(\frac{\psi}{\chi}\nu\mu)}{\log(\gamma)}$$

where $\psi = \inf\{\frac{1}{8}; \frac{1}{2\beta}\}$ and $\chi = \sup\{4C^+; 6\alpha C^-\}$ are positive constants.

2.2. Windows and shadowing

In this section, we refer to the work of R. Easton [14] and Alekseev [1] where all the machinery of windows (or topological Markov's chains) has been discussed. The main point is that, if one considers a family of maps of the unit square in a given manifold, one can proved the existence of orbits with a prescribed behaviour if these maps satisfy a condition of *alignment* and are weakely *hyperbolics.*

Windows. Let $\mathcal{M}$ be a d dimensional manifold, and d_h, d_v two integers such that $d = d_h + d_v$, we denote $I = [-1, 1]$ the unit intervall. A (d_h, d_v)-*windows* in $\mathcal{M}$ is a C^1 diffeomorphism $\mathcal{W}$ of $I^d = I^{d_h} \times I^{d_v}$ in $\mathcal{S}$. Its *horizontals* are the partial maps $\mathcal{W}(., y_v)$, for $y_v \in I^{d_v}$, and its *verticals* are the partial maps $\mathcal{W}(y_h, .)$, for $y_h \in I^{d_h}$. We denote by $\hat{\mathcal{W}}$ the image of the window $\mathcal{W}$.

Alignment. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two (d_h, d_v)-windows in $\mathcal{S}$. We say that $\mathcal{W}_1$ is *aligned* with $\mathcal{W}_2$ if the following conditions are satisfied :

(i) *Transversality condition.* Each horizontal $\mathcal{W}_1(., y_v)$ of the first window is transvers to each vertical $\mathcal{W}_2(y_h, .)$ of the second window.

(ii) *Intersection condition.* The image of $\mathcal{W}_1(., y_v)$ and $\mathcal{W}_2(y_h, .)$ intersect in a single point a such that $a = \mathcal{W}_1(x_h, y_v) = \mathcal{W}_2(y_h, x_v)$, with $(x_h, x_v) \in]-1, 1[^d$.

Of course, correct alignment of windows is stable under C^1 perturbation (see lemma 3.3 of [14]). We will use this property in the following.

Using these notions, we can prove the following version of the *shadowing lemma* (see V.M. Alekseev [1], R.W. Easton [14]):

Shadowing lemma. *Let $J \in \mathbf{N}$, and $(\mathcal{W}_j)_{1\leq j\leq J}$ be a (finite) family of windows in a manifold $\mathcal{M}$, and $(\Psi_j)_{1\leq j\leq J}$ a family of diffeomorphisms such that Ψ_j is defined in the neighbourhood of $\hat{\mathcal{W}}_j$. If the window $\Psi_j \circ \mathcal{W}_j$ is aligned with $\mathcal{W}_{j+1}$ for $1 \leq j \leq J-1$, there exists (at least) one point a of $\hat{\mathcal{W}}_1$ such that $\Psi_j \circ \ldots \circ \Psi_1(a)$ is defined for all j and belongs to $\hat{\mathcal{W}}_{j+1}$.*

In our case, the manifold $\mathcal{M}$ will be the set of all the Poincaré sections that we have chosen in the neighbourhood of each torus, and the diffeomorphism Ψ_j will be the map $F_{j+1}^{n_j} \circ \Theta_j^{-1}$ for a suitable $n_j \in \mathbb{N}$.

Our problem is to choose the windows $\mathcal{W}_j$ such that the alignment condition is fullfilled with a suitable choice for n_j.

Alignment criterion. First, we recall a criterion for correct alignment of *linear* windows due to Easton [14].

Let W_A and W_B be two (d_h, d_v) linear windows into $\mathbf{R}^d$, given by

$$W_A(x) = a + Ax, \qquad W_B(x) = b + Bx,$$

with $a \in \mathbf{R}^d$, $b \in \mathbf{R}^d$ and

$$A = \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix},$$

where $A_1 \in M_{(d_h,d_h)}(\mathbf{R})$, $A_2 \in M_{(d_h,d_v)}(\mathbf{R})$, $A_3 \in M_{(d_v,d_h)}(\mathbf{R})$ and $A_4 \in M_{(d_v,d_v)}(\mathbf{R})$. Similarly we write B.

We define two *intermediary* matrices given by

$$M = \begin{pmatrix} A_1 & -B_3 \\ A_2 & -B_4 \end{pmatrix}, \qquad N = \begin{pmatrix} B_1 & -A_3 \\ B_2 & -A_4 \end{pmatrix}.$$

Then, a necessary condition for correct alignment of linear windows is given by the following lemma.

LEMMA 2.1. *Let M and N be the two previous intermediary matrices associated to linear windows W_A and W_B. If*

i) M is non singular,
ii) $\| M^{-1}(b-a) + M^{-1}Ny \|_\infty < 1$ for all $y \in I^d$,

then W_A correctly aligns with W_B.

We refer to Easton [14] or Marco [20] for a proof.

LEMMA 2.2. *Let $\mathcal{W}_A$ and $\mathcal{W}_B$ two (d_h, d_v) windows of the form*

$$\mathcal{W}_A(x) = a + Ax + \hat{A}(x), \qquad \mathcal{W}_B(x) = b + Bx + \hat{B}(x)$$

where $a, b \in \mathbf{R}^d$*,* A*,* B *are in* $\mathbf{Gl}_d(\mathbf{R})$ *and* $\hat{\mathcal{A}}$*,* $\hat{\mathcal{B}}$ *are application of class* C^2 *. Assume that the linear windows* $\mathcal{W}_A^l : x \to a + Ax$ *and* $\mathcal{W}_B^l : x \to b + Bx$ *are correctly aligned.*

Let M *and* N *be the two intermediary associated matrices,* $\beta = \| M^{-1} \|_1$ *and*

$$\kappa = \sup_{y \in L^d} \| M^{-1}(b-a) + M^{-1}Ny \|_\infty \quad (< 1)$$

such that

(i) $\beta(\| \hat{\mathcal{A}} \|_{(2)} + \| \hat{\mathcal{B}} \|_{(2)}) = \kappa_1 < 1/3$*,*
(ii) $\beta(\| D\hat{\mathcal{A}} \|_{(2)} + \| D\hat{\mathcal{B}} \|_{(2)}) = \kappa_2$*, where* $\kappa_2 < 1$ *and* $\dfrac{\kappa_2}{1-\kappa_2} < 1/4$*,*
(iii) $\kappa + \kappa_1 < 1$*,*

then $\mathcal{W}_A$ *and* $\mathcal{W}_B$ *are correctly aligned.*

2.3. Proof of theorem 2.1

Let $(S, U) \in \mathbb{R}^m \times \mathbb{R}^m$, and $\mathcal{B}$ the ball defined by

$$\mathcal{B} = \{(S,U) \in \mathbb{R}^m \times \mathbb{R}^m; \| S \|_\infty \le 1, \| U \|_\infty \le 1\}$$

Following the scheme given by Easton [14], we construct windows $(\mathcal{W}_i)_{1 \le i \le N} : \mathcal{B} \to B_i^-$ of the form

$$\mathcal{W}_i(x) = b_i + B_i x + \hat{B}_i(x),$$

where the center $b_i \in B_i^-$ is near p_i^-; such that for a given n, the window $F_{i+1}^n \circ \Theta_i^{-1} \circ \mathcal{W}_i$ is correctly aligned with the window $\mathcal{W}_{i+1}$.

We first consider *linear windows* $(\mathcal{L}_i) : \mathcal{B} \to B_i^+$, of the form

$$\mathcal{L}_i(x) = l_i + L_i.x,$$

where the center l_i is near p_i^+. We can choose the windows $\mathcal{W}_i$ to be

$$\mathcal{W}_i = \Theta_i \circ \mathcal{L}_{i+1}.$$

We denote

$$\mathcal{T}_i(x) = \Psi_i \circ \mathcal{W}_i,$$

where

$$\Psi_i = F_{i+1}^n \circ \Theta_i^{-1}$$

and

$$\mathcal{T}_i(x) = a_i + A_i.x + \hat{A}_i(x).$$

We have the following lemma, to be proved in the next section.

LEMMA 2.3. *For l_i and L_i of the form*

$$L_i = \frac{\mu}{2} \begin{pmatrix} 2\psi I & -\Lambda_i^n \sigma_{i-1}^{-1} a_{i-1} \\ 0 & \Lambda_i^n \end{pmatrix}, \tag{7}$$

and $l_i = (s_i^+ - a_{i-1}\sigma_{i-1}^{-1} u_i, u_i)$, with $\| \Lambda_i^{-n} u_i - u_i^- \|_\infty < \frac{\mu}{2}$, the windows $\mathcal{T}_i$ and $\mathcal{W}_{i+1}$ are correctly aligned for μ sufficiently small and $i = 1, \dots, N-1$, if n satisfies the following constraint :

$$\gamma^n < \frac{\psi}{\chi} \nu \mu \tag{8}$$

where $\chi = \sup\{4C^+; 6\alpha C^-\}$ and $\psi = \inf\{\frac{1}{8}; \frac{1}{2\beta}\}$.

The choice of l_i and L_i is explained in §.2.4. In §.2.5, we prove that $\mathcal{T}_i^l = a_i + A_i x$ and $\mathcal{W}_{i+1}^l = b_{i+1} + B_{i+1} x$ are correctly aligned for this choice of n. Then, using the alignment criterion, we deduce the lining up of the windows $\mathcal{T}_i$ and $\mathcal{W}_{i+1}$.

Proof of theorem 2.1. The windows $\mathcal{T}_i = \Psi_i \circ \mathcal{W}_i$ and $\mathcal{W}_{i+1}$ are correctly aligned for n as in lemma 2.3 and $i = 1, \dots, N(\mu) - 1$ for all μ sufficiently small. We deduce from the shadowing lemma that there exists (at least) one point a of B_1^- such that $\Psi_{N(\mu)-1} \circ \dots \circ \Psi_1(a)$ is defined and belongs to $B_{N(\mu)}^-$. We denote by $\zeta(t)$ the associated connecting orbit, and T the time such that $\zeta(0) \in B_1^-$ and $\zeta(T) \in B_{N(\mu)}^-$. The time T can be expressed as follows

$$T = C(\mathcal{P}) N(\mu) \tau(\mu) n,$$

where $\tau(\mu)$ is the period of each orbit, $C(\mathcal{P}) > 0$ is a constant and n is such that (8) is satisfied. We can choose

$$n = \frac{\log(\frac{\psi}{\chi}\nu\mu)}{\log(\gamma)}.$$

This conludes the proof of theorem 2.1. □

2.4. Window map

We denote by $\mathcal{L}_i(x) = l_i + L_i.x$, with $L_i = \begin{pmatrix} \hat{a}_i & \hat{b}_i \\ \hat{c}_i & \hat{d}_i \end{pmatrix}$, the window map. We choose the matrix L_i and a point l_i such that the following constraints are satisfied :

(i) For $x \in \mathcal{B}$, we have $\mathcal{L}_i(x) \in B_i^+$
(ii) The horizontals of $\mathcal{L}_i$ are parallel to the stable manifold of the periodic orbit O_i
(iii) The image of the verticals of $\mathcal{L}_{i+1}$ by Θ_i^l are parallel to the unstable manifold of the periodic orbit O_i
(iv) $\Theta_i^l(l_{i+1}) \in W^-(O_i)$ and for $x \in \mathcal{B}$, we have $\Theta_i^l \circ \mathcal{L}_{i+1}(x) \in B_i^-$
(v) For $x \in \mathcal{B}$, and $n \in \mathbb{N}$ well choosen, we have $(F_i^l)^n \circ \mathcal{L}_i(x) \in B_i^-$.

We then have the following lemma :

LEMMA 2.4. *A matrix L_i and a point l_i such that the geometric conditions (i), (ii), (iii) and (iv) are satisfied by the window map $\mathcal{L}_i$, are given by*

$$L_i = \begin{pmatrix} \hat{a}_i & -\delta_i a_{i-1} \\ 0 & \delta_i \sigma_{i-1} \end{pmatrix},$$

and $l_i = (s_i^+ - a_{i-1}\sigma_{i-1}^{-1}u_i, u_i)$, with $\| \Lambda_i^{-n} u_i - u_i^- \|_\infty < \dfrac{\mu}{2}$ and the following constraints to choose n, δ_i and $\hat{a}_i$:

$$\begin{array}{l} \gamma^n < \dfrac{\psi}{6\alpha C^-}\nu\mu \\ \| \hat{a}_i \| < \psi\mu \\ \| \Lambda_i^{-n}\delta_i\sigma_{i-1} \| < \frac{\mu}{2} \end{array} \tag{9}$$

where $\psi = \inf\{\frac{1}{8}; \frac{1}{2\beta}\}$.

Proof of Lemma 2.4. Let $h_{U_0} : S \to \mathcal{L}_i(S, U_0)$ be a horizontal of $\mathcal{L}_i$. Condition ii) implies that $\hat{c}_i S + \hat{d}_i U_0 = \text{const}$, for all $S \in \mathcal{B}$. We deduce that $\hat{c}_i = 0$. Then, L_i is of the form $L_i = \begin{pmatrix} \hat{a}_i & \hat{b}_i \\ 0 & \hat{d}_i \end{pmatrix}$.

Let $v_{S_0} : U \to \mathcal{L}_{i+1}(S_0, U)$ be a vertical of $\mathcal{L}_{i+1}$. Condition iii) gives the following constraint on L_{i+1} :

$$\sigma_i \hat{b}_{i+1} + a_i \hat{d}_{i+1} = 0.$$

We then take (for example) :

$$\begin{array}{rcl} \hat{b}_{i+1} & = & -\delta_{i+1} a_i, \\ \hat{d}_{i+1} & = & \delta_{i+1}\sigma_i, \end{array}$$

where δ_{i+1} is a **matrix** which controls the size of the window $\mathcal{L}_{i+1}$.

We denote $(s, u) = \mathcal{L}_i(S, U)$ for $(S, U) \in \mathcal{B}$. We have

$$\begin{array}{rcl} s & = & s_i + \hat{a}_i S - \delta_i a_{i-1} U, \\ u & = & u_i + \delta_i \sigma_{i-1} U. \end{array}$$

By i), we must have

$$\begin{array}{lll} \| s_i - s_i^+ + \hat{a}_i S - \delta_i a_{i-1} U \|_\infty & < \mu, & \text{(i.1)}, \\ \| u_i + \delta_i \sigma_{i-1} U \|_\infty & < \mu, & \text{(i.2)}. \end{array}$$

By iv), we must have $\Theta_i^l(l_{i+1}) \in W^-(O_i)$, but

$$\Theta_i^l(l_{i+1}) = (\sigma_i(s_{i+1} - s_{i+1}^+) + a_i u_{i+1}, b_i(s_{i+1} - s_{i+1}^+) + u_{i+1} + u_i^-).$$

Condition iv) implies that $\sigma_i(s_{i+1} - s_{i+1}^+) + a_i u_{i+1} = 0$, which gives

$$s_{i+1} = s_{i+1}^+ - \sigma_i^{-1} a_i u_{i+1}. \tag{10}$$

Moreover, by iv) we must have

$$\begin{array}{ll} \| \sigma_i(s_{i+1} - s_{i+1}^+) + a_i u_{i+1} + \sigma_i \hat{a}_{i+1} S \|_\infty & < \mu, \\ \| b_i(s_{i+1} - s_{i+1}^+) + u_{i+1} + b_i \hat{a}_{i+1} S + \delta_{i+1}(\sigma_i - a_i b_i) U \|_\infty & < \mu, \end{array}$$

As we have (10), condition iv) can be written as :

$$\begin{array}{lll} \| \sigma_i \hat{a}_{i+1} S \|_\infty & < \mu, & \text{(iv.1)}, \\ \| (I - \sigma_i^{-1} a_i b_i)(u_{i+1} + \delta_{i+1} \sigma_i U) + b_i \hat{a}_{i+1} S \|_\infty & < \mu, & \text{(iv.2)}. \end{array}$$

By v),we must have

$$\begin{array}{lll} \| \Lambda_i^n s_i + \Lambda_i^n \hat{a}_i S - \Lambda_i^n \delta_i a_{i-1} U \|_\infty & < \mu, & \text{(v.1)}, \\ \| \Lambda_i^{-n} u_i - u_i^- + \Lambda_i^{-n} \delta_i \sigma_{i-1} U \|_\infty & < \mu, & \text{(v.2)}. \end{array}$$

We assume that

$$\| \Lambda_i^{-n} u_i - u_i^- \|_\infty < \frac{\mu}{2} \tag{11}$$

Then the constraint (9) on δ_i implies that the condition (v.2) is verified. As we have the constraint (9) on n, it is sufficient to prove the condition (i.1) to obtain the condition (v.1) for μ sufficiently small .

By (11), for μ sufficiently small, we have

$$\| u_i \|_\infty < \frac{3}{2} \gamma^n C^- \tag{12}$$

Otherwise, by (10), we have

$$\| s_i - s_i^+ \|_\infty < \nu^{-1} \alpha \| u_i \|_\infty \tag{13}$$

So the constraint (9) on n implies $\| s_i - s_i^+ \|_\infty < \frac{\psi\mu}{4}$ and $\| \delta_i a_{i-1} \| < \frac{\mu}{8}$. Moreover, by the constraint (9) on $\hat{a}_i$, the condition (i.1) is verified.

By the constraint (9) on δ_i and by (12), we obtain the condition (i.2). Then, by the constraint (9) on $\hat{a}_i$, we deduce that the condition (iv.2) is verified.

Finally, for μ sufficiently small, the constraint (9) on $\hat{a}_i$ implies that the condition (iv.1) is verified. This concludes the proof of the lemma 2.4. □

2.5. Proof of Lemma 2.3

a. Intermediary matrices. We choose $\hat{a}_i$, δ_i and n satisfying the constraints (9) to have the alignment of the linear windows $\mathcal{W}_{i+1}^l$ and $\mathcal{T}_i^l$. We can take $\hat{a}_i$ independent of i. Hence, in the subsequent of the paper, we denote $\hat{a}_i$ by $\hat{a}$ which will be defined by

$$\hat{a} = \psi\mu I.$$

Otherwise, δ_i will be choosen dependent of i by

$$\delta_i = \frac{\mu}{2} {\Lambda_i}^n {\sigma_{i-1}}^{-1}.$$

Then, using the lemma 2.4, we compute the matrix B_{i+1} and A_i of the linear windows $\mathcal{W}_{i+1}^l$ and $\mathcal{T}_i^l$:

$$B_{i+1} = \frac{\mu}{2} \begin{pmatrix} 2\psi\sigma_{i+1} & 0 \\ 2\psi b_{i+1} & \Lambda_{i+2}^n \sigma_{i+1}^{-1}(\sigma_{i+1} - a_{i+1} b_{i+1}) \end{pmatrix}$$

$$A_i = \frac{\mu}{2} \begin{pmatrix} 2\psi\Lambda_{i+1}^n & -\Lambda_{i+1}^{2n} \sigma_i^{-1} a_i \\ 0 & I \end{pmatrix}.$$

with $b_{i+1} = (0, u_{i+2}(I - \sigma_{i+1}^{-1} a_{i+1} b_{i+1}) + u_{i+1}^-)$, and $a_i = (\Lambda_{i+1}^n(s_{i+1}^+ - \sigma_i^{-1} a_i u_{i+1}), \Lambda_{i+1}^{-n} u_{i+1})$.

We deduce that the intermediary matrices are given by

$$M_i = \frac{\mu}{2} \begin{pmatrix} 2\psi\Lambda_{i+1}^n & 0 \\ 0 & -\Lambda_{i+2}^n \sigma_{i+1}^{-1}(\sigma_{i+1} - a_{i+1} b_{i+1}) \end{pmatrix},$$

$$N_i = \frac{\mu}{2} \begin{pmatrix} 2\psi\sigma_{i+1} & \Lambda_{i+1}^{2n} \sigma_i^{-1} a_i \\ 2\psi b_{i+1} & -I \end{pmatrix}.$$

For μ sufficiently small we have $det(\sigma_{i+1} - a_{i+1}b_{i+1}) \neq 0$, and M_i is invertible. We then have :

$$M_i^{-1} = 2\mu^{-1} \begin{pmatrix} (2\psi)^{-1}\Lambda_{i+1}^{-n} & 0 \\ 0 & -(\sigma_{i+1} - a_{i+1}b_{i+1})^{-1}\sigma_{i+1}\Lambda_{i+2}^{-n} \end{pmatrix},$$

and

$$M_i^{-1}N_i = \begin{pmatrix} \Lambda_{i+1}^{-n}\sigma_{i+1} & (2\psi)^{-1}\Lambda_{i+1}^{n}\sigma_i^{-1}a_i \\ -2\psi(\sigma_{i+1} - a_{i+1}b_{i+1})^{-1}\sigma_{i+1}\Lambda_{i+2}^{-n}b_{i+1} & (\sigma_{i+1} - a_{i+1}b_{i+1})^{-1}\sigma_{i+1}\Lambda_{i+2}^{-n} \end{pmatrix}.$$

b. Estimates. We have

$$b_{i+1} - a_i = (-\Lambda_{i+1}^{n}(s_{i+1}^{+} - \sigma_i^{-1}a_iu_{i+1}), u_{i+2}(I - \sigma_{i+1}^{-1}a_{i+1}b_{i+1}) + u_{i+1}^{-} - \Lambda_{i+1}^{-n}u_{i+1}).$$

We choose

$$u_{i+1} = \Lambda_{i+1}^{n}(u_{i+1}^{-} + u_{i+2}(I - \sigma_{i+1}^{-1}a_{i+1}b_{i+1})).$$

Then we have

$$\| u_i \|_{\infty} < C_1\gamma^n, \tag{14}$$

for all $i = 1, \ldots, N(\mu)$. Moreover, as the windows $\mathcal{L}_i$ are constructed recursively, it is sufficient to assume that $u_{N(\mu)}$ is such that (11) is satisfied to have

$$\| \Lambda_i^{-n}u_i - u_i^{-} \|_{\infty} < \frac{\mu}{2},$$

for all $i = 1, \ldots, N(\mu) - 1$.

We deduce that

$$M_i^{-1}(b_{i+1} - a_i) = (-(\psi\mu)^{-1}(s_{i+1}^{+} - \sigma_i^{-1}a_iu_{i+1}), 0)$$

We denote $M_i^{-1}N_i(S,U) + M_i^{-1}(b_{i+1} - a_i) = (S', U')$. We have

$$\begin{aligned} S' &= \Lambda_{i+1}^{-n}\sigma_{i+1}f(S) + (\psi\mu)^{-1}\sigma_i^{-1}a_iu_{i+1} + (2\psi)^{-1}\Lambda_{i+1}^{n}\sigma_i^{-1}a_iU, \\ U' &= (\sigma_{i+1} - a_{i+1}b_{i+1})^{-1}\sigma_{i+1}\Lambda_{i+2}^{-n}\left[-2\psi b_{i+1}f(S) + g(U)\right]. \end{aligned}$$

where

$$\begin{aligned} f(S) &= S - (\psi\mu)^{-1}\sigma_{i+1}^{-1}\Lambda_{i+1}^{n}s_{i+1}^{+}, \\ g(U) &= U - 2\mu^{-1}b_{i+1}\sigma_{i+1}^{-1}\Lambda_{i+1}^{n}s_{i+1}^{+}. \end{aligned}$$

We want to prove that $\| S' \|_\infty < 1$, $\| U' \|_\infty < 1$ for all $(S,U) \in \mathcal{B}$. We define the set $\mathcal{V}_n$ by

$$\begin{array}{l} \| f(S) \| < \frac{\xi\sigma^{-1}\rho^n}{3} \\ \| g(U) \| < \xi\sigma^{-1}\rho^n \end{array}$$

where $\xi = \inf\{1; \frac{1}{4K}\}$.

We choose n satisfying the constraint (8) on n. Then $\mathcal{V}_n \subset \mathcal{B}$. Moreover, the constraint (9) on n is verified and for $(S,U) \in \mathcal{V}_n$ and μ sufficiently small, we have easily $\| S' \| < 1$ and $\| U' \| < 1$.

d. Remainder of $\mathcal{W}_{i+1}$ windows. We have

$$\mathcal{W}_{i+1}(x) = \Theta_{i+1} \circ \mathcal{L}_{i+2}(x) = b_{i+1} + B_{i+1}x + \hat{B}_{i+1}(x).$$

As we can write

$$\Theta_{i+1}(\mathcal{L}_{i+2}(x)) = \Theta_{i+1}(l_{i+2}+L_{i+2}x) = \Theta_{i+1}(l_{i+2})+D_{l_{i+2}}\Theta_{i+1}(L_{i+2}x)+R(x),$$

with

$$R(x) = \int_0^1 \frac{(1-t^2)}{2} D^2\Theta_{i+1}(l_{i+2} + tL_{i+2}x)(L_{i+2}x)^2 dt,$$

and as we have $b_{i+1} = \Theta_{i+1}(l_{i+2})$, we obtain

$$\hat{B}_{i+1}(x) = R(x) + (D_{l_{i+2}}\Theta_{i+1}.L_{i+2} - B_{i+1})x.$$

As $\sup_{B^+} \| D^2\Theta_{i+1} \| \leq C$ by the regularity assumption, we deduce that

$$\| R \|_{(2)} \leq 4C\mu^2.$$

Moreover, as $B_{i+1} = D_{p^+_{i+2}}\Theta_{i+1}L_{i+2}$, we obtain

$$\begin{array}{rcl} \| D_{l_{i+2}}\Theta_{i+1}.L_{i+2} - B_{i+1} \|_1 & = & \| (D_{l_{i+2}}\Theta_{i+1} - D_{p^+_{i+2}}\Theta_{i+1})L_{i+2} \|_1 \\ & \leq & C\mu \| l_{i+2} - p^+_{i+2} \|_1 \end{array}$$

As we have $\| l_{i+2} - p^+_{i+2} \|_1 \leq (\| s_{i+2} - s^+_{i+2} \|_1 + \| u_{i+2} \|_1)$ and by (13),(14), we deduce that

$$\| \hat{B} \|_{(2)} \leq C_3\mu(\mu + \gamma^n).$$

In the same way, we obtain

$$\| D\hat{B} \|_{(2)} \leq C_3(\mu + \gamma^n).$$

e. Remainder of $\mathcal{T}_i$ windows. We have $\mathcal{T}_i(x) = a_i + A_i x + \hat{A}_i(x)$, with

$$\hat{A}_i(x) = r_n(l_i + L_i x).$$

By (14), we have $\parallel u_i \parallel_1 \leq C_3\gamma^n$. Then for all $y = (u(y); s(y)) \in \mathcal{L}_i(\mathcal{B})$, we have $\parallel u_y \parallel_1 \leq C_4\gamma^n$ and as $\parallel s_y \parallel_1 \leq 1$, we have $\parallel s_y u_y \parallel_1 \leq C_4\gamma^n$. We then prove that

$$\parallel \hat{A}_i \parallel_{(2)} \leq C_5\gamma^n,$$

and

$$\parallel D\hat{A}_i \parallel_{(2)} \leq C_6\gamma^n.$$

We then easily satisfy the alignment criterion for perturbed linear windows. This concludes the proof of lemma 2.3. □

3. ARNOLD DIFFUSION TIME

Using symbolic dynsmics, we prove, following [10], that there exists along a transition chain, a dual chain of hyperbolic periodic orbits. Using [11], we give quantitative informations (periods, angle) about this dual chain. By applying theorem 2.1, we obtain a general estimate for the Arnold diffusion time in initially hyperbolic Hamiltonian systems. In the particular case of the Arnold's example, we obtain a Arnold diffusion time of order $\frac{1}{\mu}\log\left(\frac{1}{\mu}\right)$, which has been conjectured by Lochak [18].

3.1. Graff tori and symbolic dynamics

3.1.1. Graff tori

Let (M, Ω) be a $2n$-dimensional smooth symplectic manifold and H_μ a Hamiltonian function defined on M.

A *Graff torus* is an invariant m-partially hyperbolic torus $\mathcal{T}$ for which there exists a neighbourhood $V(\mu)$ such that the Hamiltonian H_μ takes the form

$$H_\mu(\theta, I, s, u) = \omega I + \lambda su + f(I, su) + \mu g(I, \theta, s, u),$$

with $\lambda > 0$, $(\theta, I) \in \mathbb{T}^l \times \mathbb{R}^l, (s, u) \in \mathbb{R}^m \times \mathbb{R}^m$, $0 < \mu \ll 1$, $f(I, su) = O_2(I, su)$ and $g(\theta, I, s, u) = O_2(I, su; \theta, I, s, u)$.

We assume that ω satisfies a Diophantine condition

$$|\ \omega.k\ | \geq \gamma |\ k\ |^{-\tau},$$

for all $k \in \mathbb{Z}^l \setminus \{0\}$, with $\gamma > 0$ and $\tau > n-1$.

Moreover, we assume that $V(\mu)$ is of the form

$$V(\mu) = \{(\theta, I, s, u) \in \mathbb{T}^l \times \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^m | \parallel s \parallel < \kappa, \parallel U \parallel < \kappa, \parallel I \parallel < \kappa\mu\},$$

3.1.2. Poincaré section

As ω is non resonant, there exists a surface of section S and an analytic coordinates system $(\theta, \rho, s, u) \in \mathbb{T}^{l-1} \times \mathbb{R}^{l-1} \times \mathbb{R}^m \times \mathbb{R}^m$ in which the Graff torus $\mathcal{T}$ is given by

$$\mathcal{T} = \{(\theta, \rho, s, u) \in \mathbb{T}^{l-1} \times \mathbb{R}^{l-1} \times \mathbb{R}^m \times \mathbb{R}^m | \rho = 0, s = u = 0\},$$

and the associated Poincaré, denoted by f, has the form

$$f(\theta, \rho, s, u) = (\theta + v(\rho), \rho, \lambda s, \lambda^{-1} u) + r(\theta, I, s, u),$$

where $\lambda < 1$ and r is of order 2 in ρ, s, u in a domain

$$V(\mu) = \{(\theta, \rho, s, u) \in \mathbb{T}^{l-1} \times \mathbb{R}^{l-1} \times \mathbb{R}^m \times \mathbb{R}^m | \parallel s \parallel < \kappa, \parallel u \parallel < \kappa, \parallel \rho \parallel < \kappa\mu\},$$

Moreover, we have $v(\rho) = \nu + \nu_1 \rho$ where ν satisfies a diophantine condition

$$| \nu.k | \geq \gamma | k |^{-\tau},$$

for all $k \in \mathbb{Z}^{l-1} \setminus \{0\}$ and $\nu_1 \neq 0$.

3.1.3. Symbolic dynamics and Graff tori

Let $\mathcal{H} = H^{-1}(h)$ be the energy level containing the invariant torus $\mathcal{T}$ and its stable and unstable manifolds.

Assumptions . It is assumed that:

(h_1) The stable and unstable manifolds of $\mathcal{T}$ intersect transversally in $\mathcal{H}$ along (at least) a homoclinic orbit Γ.

The homoclinic orbit Γ intersects the section S in $p^+ = (\theta^+, s^+, 0, 0) \in W^+(T)$ and $p^- = (\theta^-, 0, 0, u^-) \in W^-(T)$. We define two neighbourhoods of p^+ and p^- as

$$B^+ = \{z \in V(\mu) | \| \theta - \theta^+ \| < \delta^+, \| s - s^+ \| < \delta^+, \| \rho \| < \delta^+ \mu, \| u \| < \delta^+\},$$
$$B^- = \{z \in V(\mu) | \| \theta - \theta^- \| < \delta^-, \| s \| < \delta^-, \| \rho \| < \delta^- \mu, \| u - u^- \| < \delta^-\},$$

where $\delta^+ > 0$ and $\delta^- > 0$ are sufficiently small, independent of μ, such that $B^+ \subset V(\mu)$ and $B^- \subset V(\mu)$.

Transversal map . Let $\tau : S \to [0, \infty[$,

$$\tau(p) = \{t > 0 | \Phi(t, p) \in \mathcal{H} \setminus S \text{ for } 0 < s \leq t\},$$

where Φ is the flow associated to the Hamiltonian vector field X_H.

We define the set $\Xi = \{p \in S | \tau(p) < \infty\}$. We can suppose δ^+ sufficiently small to have $B^+ \subset \Xi$ and $B^- \subset f(\Xi)$. We denote

$$D_n = \{z \in B^+ | f^n(z) \in B^-\}.$$

We also choose δ^+ and δ^- sufficiently small such that $D_n \cap D_m \neq \emptyset$ if $n \neq m$. We denote $D = \bigcup_{n \geq 1} D_n$.

Then the *transversal map* $\mathcal{F} : D \to B^-$ is defined by

$$\mathcal{F}(z) = f^n(z), \forall z \in D_n.$$

Homoclinic map . The *homoclinic map* Λ is defined by

$$\Lambda(q) = \Phi(q, \sigma(q))$$

where $\sigma(q) = \inf\{t > 0 | \Phi(t, q) \in B^+\}$.

$\Lambda : B^- \to B^+$ has the form

$$\Lambda(z) = p^+ + \Pi h + \Lambda_2 h$$

where $h = z - p^-$, $\Pi = D_{p^-}(\Lambda)$ and Λ_2 is the term of order ≥ 2.

(h_2) We assume that Π is writen in the basis $(e_\theta, e_s, e_\rho, e_u)$ takes the form

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ \alpha & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

where $(a, b, c, d) \in \mathbb{R}^4$ is such that $ad - bc \neq 1$, $\alpha \neq 0$ is of order μ^k with $k > 0$.

(h_3) We assume that

$$\Lambda_2 = O_2(I, su).$$

Transition map . We denote $\Psi : B^+ \to B^+$ the *transition map* defined by

$$\Psi = \Lambda \circ F$$

Alphabet. We define the *alphabet* set by

$$\mathcal{A} = \{n \in \mathbb{N}, n_0 \leq n < \infty, \mid \theta^+ - \theta^- + n\nu \mid < C\mu^{1+2\delta}\},$$

where $C_2 \in \mathbb{R}$ is a constant and $n_0 = \alpha\mu^{-(1+2\delta)\tau}$ with $\delta > 0$ an arbitrary constant.

Window . Let $X = (\Theta, S) \in \mathbb{T} \times \mathbb{R}, Y = (R, U) \in \mathbb{R} \times \mathbb{R}$, and $\mathcal{B}$ the ball of center O and radius 1. We denote for $n \in \mathcal{A}$,

$$\mathcal{H}_n = \{Z = (\Theta, S, R, U) \in \mathbb{T}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}, \mid U - \lambda^n\mu^{-1}u^- \mid < \lambda^n, \mid R \mid < \lambda^n\},$$

and $\mathcal{D} = \bigcup_{n\in\mathcal{A}} \mathcal{H}_n$. We define the window $\mathcal{W} : \mathcal{B} \to \mathcal{B}^+$ by $\mathcal{W}(Z) = p^+ + W.Z$ with

$$W = \begin{pmatrix} \mu & 0 & \mu^{\kappa+1}\alpha^{-1} & 0 \\ 0 & \mu & 0 & b\mu d^{-1} \\ 0 & 0 & \mu^{\kappa+1} & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}$$

with $\kappa > 0$ such that $\mu^\kappa\alpha^{-1} = \mu^\delta$ where $\delta > 0$.

We denote by $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ the *window map* defined by $\mathcal{L} = \mathcal{W}^{-1} \circ \Psi \circ \mathcal{W}$.

Then we have the following theorem, proved in [11]:

THEOREM 3.2. *Under the hypotheses (h_1), (h_2), (h_3), and for μ sufficiently small, the map $\mathcal{L}$ possesses an hyperbolic invariant set*

$$I = \{Z \in \mathcal{D} | \mathcal{L}^n(Z) \in \mathcal{D}, \forall n \in \mathbb{N}\},$$

such that $\mathcal{L}|_I$ is topologically conjugated to the shift on the alphabet $\mathcal{A}$. Then, there exists a homeomorphism ϕ such that the following diagram commutes

$$\begin{array}{ccc} I & \longmapsto & I \\ \phi \downarrow & & \downarrow \phi \\ \Sigma^{\mathcal{A}} & \longmapsto & \Sigma^{\mathcal{A}} \end{array}$$

where $\Sigma^{\mathcal{A}} = \prod_{i=-\infty}^{+\infty} \mathcal{A}$ *and σ is the shift mapping on this space.*

3.2. Transition chain and hyperbolic orbits

Let $\mathcal{T} = \{T_1, \dots, T_N\}$ be a family of Graff tori for H_μ such that $W^+(T_i)$ and $W^-(T_{i+1})$ intersect transversally in a given energy manifold $\mathcal{H}$. Then $\mathcal{T}$ is a transition chain (see [20],[8]).

Let T_i be a torus of $\mathcal{T}$. We denote by S_i its surface of section, h_i a homoclinic orbit to T_i. We denote by p_i^+ (resp. p_i^-) the intersection of h_i with $W^+(T_i) \cap S_i$ (resp. $W^-(T_i) \cap S_i$) and B_i^+ (resp. B_i^-) a neighbourhood of p_i^+ (resp. p_i^-) .

For all $i = 1, \dots, n$ we denote Ψ_i the homoclinic map, $\mathcal{W}_i$ the window, $l_i = \mathcal{W}_i^{-1} \circ \Psi_i \circ \mathcal{W}_i$, ϕ_i the homeomorphism of the theorem 3.2 defined in the neighbourhood of T_i, $\mathcal{A}_i$ the alphabet associated and I_i the invariant hyperbolic set.

For $n \in \mathcal{A}_i$, we denote by (n) the infinite sequence $\{\dots, n, n, n, \dots\}$. Let $p_i(n) = \phi_i^{-1}(n)$ be the fixed point of l_i in I_i by ϕ_i^{-1}. We have

$$\Psi_i \circ \mathcal{W}_i(p_i(n)) = \mathcal{W}_i(p_i(n)).$$

Then $\mathcal{W}_i(p_i(n))$ is a fixed point of Ψ_i. Moreover, we deduce that the orbit through $\mathcal{W}_i(p_i(n))$ is periodic of period n. We denote by O_n^i this periodic orbit.

Lemma 3.5. *For all i, the periodic orbit O_n^i is hyperbolic and its stable manifold (resp. unstable) denoted by $W^+(O_n^i)$ (resp. $W^-(O_n^i)$) is $(m+1)$ dimensional.*

Let γ_i be a heteroclinic orbit between the tori T_i and T_{i+1} and $\beta_{i,i+1}$: $B_i^- \to B_{i+1}^+$ the heteroclinic map.

Let T be a torus of the family $\mathcal{T}$. We denote $w_i^{\xi,\sigma}(T) = W^{\xi}(T) \cap B_i^{\sigma}$ for $\xi = \pm$ and $\sigma = \pm$.

We assume that:
(*) the orbit γ_i intersects B_i^- (resp. B_{i+1}^+) in $q_i^- \in W^-(T_i)$ (resp. $q_i^+ \in W^+(T_{i+1})$),
(**) there exists a diffeomorphism $\mathcal{F}_i^{\sigma} : w_i^{+,\sigma}(T_{i+1}) \to w_i^{-,\sigma}(T_i)$, for $\sigma = \pm$.

Proposition 3.1. *Under the hypotheses (*) and (**) on a given (μ,τ) transition chain $\mathcal{T}$, there exists a (μ,τ,ν_{orb}) chain of hyperbolic periodic orbits along $\mathcal{T}$, with*

$$\begin{array}{lll} \nu_{orb} & \sim & \sigma, \\ \gamma & \sim & \tau\sigma\nu_1. \end{array} \tag{15}$$

3.3. Arnold diffusion time

3.3.1. Main result

Theorem 3.3. *The time of drift along an (μ,τ) transition chain is*

$$T = C(\mathcal{P})N(\mu)\tau(\mu)\frac{\log(\sigma\mu)}{\log(\sigma\tau(\mu)\nu_1)}, \tag{16}$$

Proof. . By the theorem 2.1, we have

$$T = c(\mathcal{P})N(\mu)\tau(\mu)\frac{\log(\frac{\psi}{\chi}\nu\mu)}{\log(\gamma)},$$

By the proposition 3.1, we have $\nu_{orb} \sim \sigma$ and $\gamma \sim \sigma\tau(\mu)\nu_1$. Then we obtain

$$T \sim C'(\mathcal{P})N(\mu)\tau(\mu)\frac{\log(\sigma\mu)}{\log(\sigma\tau(\mu)\nu_1)}.$$

This concludes the proof. □

3.3.2. Application to the Arnold's example of diffusion

We consider the Arnold's model which is a three degrees of freedom initially hyperbolic Hamiltonian system defined by

$$H_\mu(\theta, I, q, p) = \frac{1}{2}I^2 + \frac{1}{2}p^2 + (cos(q) - 1)(1 + \mu(\sin(\theta_1) + \sin(\theta_2))$$

where $I = (I_1, I_2) \in \mathbb{R}^2$, $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$, $(q, p) \in \mathbb{R} \times \mathbb{T}$ and μ is a parameter such that $0 < \mu \ll 1$.

We denote by $h \in \mathbb{R}^+, h > 0$ a given energy and $\mathcal{H} = H_\mu^{-1}(h)$ the corresponding energy level.

For $\mu = 0$, the system is a decoupled product of a pendulum with two rotators. Hence, for $0 \leq J \leq \sqrt{2h}$, we deduce that there exists a family $\mathcal{T}_J$ of 1 - hyperbolic invariant tori for the system given by

$$\mathcal{T}_J = \{(\theta, I, q, p) \in \mathbb{R}^2 \times \mathbb{T}^2 \times \mathbb{R} \times \mathbb{T} | I_1 = J, I_2 = \sqrt{2h - J}, q = p = 0\},$$

For $\mu \neq 0$, the family $\mathcal{T}_J$ persists. In [2], Arnold prove that there exists a transition chain for μ sufficiently small and the splitting σ of the invariant manifolds of each torus of $\mathcal{T}_J$ is of order μ. By the proposition 3.1, we deduce that there exists a chain of hyperbolic periodic orbits along $\mathcal{T}_J$ and we have the following result

PROPOSITION 3.2. *The time of drift along $\mathcal{T}_J$ is given by*

$$T \sim \frac{1}{\mu}\log(\frac{1}{\mu}).$$

Proof. . By the theorem 3.3, we have

$$T = C(\mathcal{P})N(\mu)\tau(\mu)\frac{\log(\sigma\mu)}{\log(\sigma\tau(\mu)\nu_1)},$$

By the persistence lemma proved in [12], the period $\tau(\mu)$ of each periodic orbit must be choosen such that $\tau(\mu) \sim \log(\frac{1}{\mu})$. So, as we have $\sigma \sim \mu$, we obtain

$$\frac{\log(\sigma\mu)}{\log(\sigma\tau(\mu)\nu_1)} \sim 1$$

Finally, as we have $N(\mu) \sim \frac{1}{\mu}$, we can conclude the proof. □

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