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Time scale differential, integral, and variational embeddings of Lagrangian systems

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ABSTRACT

We introduce differential, integral, and variational delta embeddings. We prove that the integral delta embedding of the Euler–Lagrange equations and the variational delta embedding coincide on an arbitrary time scale. In particular, a new coherent embedding for the discrete calculus of variations that is compatible with the least-action principle is obtained.

1. Introduction

An ordinary differential equation is usually given in differential form, i.e.,

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad t \in [a, b],\ x(t) \in \mathbb{R}^n.$$

However, one can also consider the integral form of the equation:

$$x(t) = x(a) + \int_a^t f(s, x(s))ds, \quad t \in [a, b].$$

The differential form is related to dynamics via the time derivative. The integral form is useful for proving the existence and unicity of solutions or to study analytical properties of solutions.

In order to give a meaning to a differential equation over a new set (e.g., stochastic processes, non-differentiable functions, or discrete sets), one can use the differential or the integral form. In general, these two generalizations do not give the same object. In the differential case, we need to extend first the time derivative. As an example, we can look to Schwartz's distributions [1] or backward/forward finite differences in the discrete case. Using the new derivative, one can then generalize differential operators and then differential equations of arbitrary order. In the integral case, one needs to give a meaning to the integral over the new set. This strategy is for example used by Itô [2] in order to define stochastic differential equations, defining stochastic integrals first. In general, the integral form imposes fewer constraints on the underlying objects. This is already true in the classical case, where we need a differentiable function to write the differential form but only continuity or weaker regularity to give a meaning to the integral form.

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The notion of embedding introduced in [3] is an algebraic procedure providing an extension of classical differential equations over an arbitrary vector space. Embedding is based on the differential formulation of the equation. This formalism was developed in the framework of fractional equations [4], stochastic processes [3], and non-differentiable functions [5]. Recently, it has been extended to discrete sets in order to discuss discretization of differential equations and numerical schemes [6].

In this paper, we define an embedding containing the discrete as well as the continuous case using time scale calculus. We use the proposed embedding in order to define time scale extensions of ordinary differential equations both in differential and integral forms.

Of particular importance for many applications in physics and mathematics is the case of Lagrangian systems governed by an Euler–Lagrange equation. Lagrangian systems possess a variational structure, i.e., their solutions correspond to critical points of a functional, and this characterization does not depend on the coordinate system. This induces strong constraints on solutions, for example the conservation of energy for autonomous classical Lagrangian systems. That is, if the Lagrangian does not depend explicitly on the independent variable t, then the energy is constant along physical trajectories. We use the time scale embedding in order to provide an analogy of the classical Lagrangian functional on an arbitrary time scale. By developing the corresponding calculus of variations, one then obtains the Euler–Lagrange equation. This extension of the original Euler–Lagrange equation passing by the time scale embedding of the functional is called a time scale *variational embedding*. Such extensions are known under the terminology of *variational integrators* in the discrete setting.

We then have three ways to extend a given ordinary differential equation: differential, integral, or variational embedding. All these extensions are a priori different. The coherence problem introduced in [3], in the context of the stochastic embedding, consider the problem of finding conditions under which these extensions coincide. Here, we prove that the integral and variational embeddings are coherent (see Theorem 11). The result is new and interesting even in the discrete setting, providing a new form of the Euler–Lagrange difference equation (see (15)) that is compatible with the least-action principle.

2. Note on the notation used

We denote by f or $t \to f(t)$ a function, and by $f(t)$ the value of the function at point t. Throughout the text we consistently use square brackets for the arguments of operators and round brackets for the arguments of all the other types of function. Functionals are denoted by uppercase letters in calligraphic mode. We denote by D the usual differential operator and by ∂_i the operator of partial differentiation with respect to the ith variable.

3. Reminder about time scale calculus

The reader interested in calculus on time scales is refereed to the book [7]. Here, we just recall the necessary concepts and fix some notation.

A nonempty closed subset of $\mathbb{R}$ is called a time scale and is denoted by $\mathbb{T}$. Thus, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$, are trivial examples of time scales. Other examples of time scales are $[-2, 4] \bigcup \mathbb{N}$, $h\mathbb{Z} := \{hz \mid z \in \mathbb{Z}\}$ for some $h > 0$, $q^{\mathbb{N}_0} := \{q^k \mid k \in \mathbb{N}_0\}$ for some $q > 1$, and the Cantor set. We assume that a time scale $\mathbb{T}$ has the topology that it inherits from real numbers with standard topology.

The forward jump $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ for all $t \in \mathbb{T}$, while the *backward jump* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ for all $t \in \mathbb{T}$, where $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m). The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$.

Example 1. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and $\mu(t) = 0$. If $\mathbb{T} = h\mathbb{Z}$, then $\sigma(t) = t + h$, $\rho(t) = t - h$, and $\mu(t) = h$. On the other hand, if $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$ is a fixed real number, then we have $\sigma(t) = qt$, $\rho(t) = q^{-1}t$, and $\mu(t) = (q - 1)t$.

In order to introduce the definition of delta derivative, we define a new set $\mathbb{T}^\kappa$ which is derived from $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximal point M, then $\mathbb{T}^\kappa := \mathbb{T} \setminus \{M\}$; otherwise, $\mathbb{T}^\kappa := \mathbb{T}$. In general, for $r \geq 2$, $\mathbb{T}^{\kappa^r} := (\mathbb{T}^{\kappa^{r-1}})^\kappa$. Similarly, if $\mathbb{T}$ has a right-scattered minimum m, then we define $\mathbb{T}_\kappa := \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}_\kappa := \mathbb{T}$. Moreover, we define $\mathbb{T}^\kappa_\kappa := \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$.

Definition 1. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}^\kappa$ if there exists a number $\Delta[f](t)$ such that for all $\varepsilon > 0$ there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - \Delta[f](t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call $\Delta[f](t)$ the *delta derivative* of f at t, and we say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided that $\Delta[f](t)$ exists for all $t \in \mathbb{T}^\kappa$.

Example 2. If $\mathbb{T} = \mathbb{R}$, then $\Delta[f](t) = D[f](t)$, i.e., the delta derivative coincides with the usual one. If $\mathbb{T} = h\mathbb{Z}$, then $\Delta[f](t) = \frac{1}{h}(f(t + h) - f(t)) =: \Delta_+[f](t)$, where Δ_+ is the usual forward difference operator defined by the last equation. If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then $\Delta[f](t) = \frac{f(qt)-f(t)}{(q-1)t}$, i.e., we get the usual derivative of quantum calculus.

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A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points and if its left-sided limit exists at left-dense points. We denote the set of all rd-continuous functions by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ and the set of all delta differentiable functions with rd-continuous derivative by $C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. It is known (see [7, Theorem 1.74]) that rd-continuous functions possess a *delta antiderivative*, i.e., there exists a function ξ with $\Delta[\xi] = f$, and in this case the *delta integral* is defined by $\int_c^d f(t)\Delta t = \xi(d) - \xi(c)$ for all $c, d \in \mathbb{T}$.

Example 3. Let $a, b \in \mathbb{T}$ with $a < b$. If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the integral on the right-hand side is the classical Riemann integral. If $\mathbb{T} = h\mathbb{Z}$, then $\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} hf(kh)$. If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then $\int_a^b f(t)\Delta t = (1-q)\sum_{t\in[a,b)} tf(t)$.

The delta integral has the following properties:

(i) if $f \in C_{rd}$ and $t \in \mathbb{T}$, then

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t);$$

(ii) if $c, d \in \mathbb{T}$ and f and g are delta differentiable, then the following formulas of integration by parts hold:

$$\int_c^d f(\sigma(t))\Delta[g](t)\Delta t = (fg)(t)\Big|_{t=c}^{t=d} - \int_c^d \Delta[f](t)g(t)\Delta t,$$

$$\int_c^d f(t)\Delta[g](t)\Delta t = (fg)(t)\Big|_{t=c}^{t=d} - \int_c^d \Delta[f](t)g(\sigma(t))\Delta t. \tag{1}$$

4. Time scale embeddings and evaluation operators

Let $\mathbb{T}$ be a bounded time scale with $a := \min \mathbb{T}$ and $b := \max \mathbb{T}$. We denote by $C([a, b]; \mathbb{R})$ the set of continuous functions $x : [a, b] \to \mathbb{R}$. As introduced in Section 3, by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ we denote the set of all real-valued rd-continuous functions defined on $\mathbb{T}$, and by $C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ the set of all delta differentiable functions with rd-continuous derivative.

A time scale embedding is given by specifying the following.

- A mapping $\iota : C([a, b], \mathbb{R}) \to C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.
- An operator $\delta : C^1([a, b], \mathbb{R}) \to C^1_{\mathrm{rd}}(\mathbb{T}^\kappa, \mathbb{R})$, called a generalized derivative.
- An operator $J : C([a, b], \mathbb{R}) \to C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, called a generalized integral operator.

We fix the following embedding.

Definition 2 (*Time Scale Embedding*)**.** The mapping ι is obtained by restriction of functions to $\mathbb{T}$. The operator δ is chosen to be the Δ derivative, and the operator J is given by the Δ integral as follows:

$$\delta[x](t) := \Delta[x](t), \qquad J[x](t) := \int_a^{\sigma(t)} x(s)\Delta s.$$

Definition 3 (*Evaluation Operator*)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We denote by $\widetilde{f}$ the operator associated to f and defined by

$$\widetilde{f} : \begin{array}{ccc} C(\mathbb{R}, \mathbb{R}) & \longrightarrow & C(\mathbb{R}, \mathbb{R}) \\ x & \mapsto & \widetilde{f}[x] := t \to f(x(t)). \end{array} \tag{2}$$

The operator $\widetilde{f}$ given by (2) is called the *evaluation operator* associated with f.

The definition of evaluation operator is easily extended in various ways. We give in Definition 4 a special evaluation operator that naturally arises in the study of problems of the calculus of variations and respective Euler–Lagrange equations (see Section 5).

Definition 4 (*Lagrangian Operator*)**.** Let $L : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a C^1 function defined for all $(t, x, v) \in [a, b] \times \mathbb{R}^2$ by $L(t, x, v) \in \mathbb{R}$. The *Lagrangian operator* $\widetilde{L} : C^1([a, b], \mathbb{R}) \to C^1([a, b], \mathbb{R})$ associated with L is the evaluation operator defined by $\widetilde{L}[x] := t \to L(t, x(t), D[x](t))$.

We consider ordinary differential equations of the form

$$O[x](t) = 0, \quad t \in [a, b],$$

where $x \in C^n(\mathbb{R}, \mathbb{R})$ and O is a differential operator of order n, $n \geq 1$, given by

$$O = \sum_{i=0}^{n} \widetilde{a_i} \cdot (D^i \circ \widetilde{b_i}), \tag{3}$$

where $(\widetilde{a_i})$ (respectively, $(\widetilde{b_i})$) is the family of evaluation operators associated to a family of functions (a_i) (respectively, (b_i)), and D^i is the derivative of order i, i.e., $D^i = \frac{d^i}{dt^i}$. Differential operators of form (3) play a crucial role when dealing with Euler–Lagrange equations.

We are now ready to define the time scale embedding of evaluation and differential operators.

Definition 5 (*Time Scale Embedding of Evaluation Operators*)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $\widetilde{f}$ the associated evaluation operator. The time scale embedding $\widetilde{f}_{\mathbb{T}}$ of $\widetilde{f}$ is the extension of $\widetilde{f}$ to $C_{rd}(\mathbb{T}, \mathbb{R})$:

$$\widetilde{f}_{\mathbb{T}} : \begin{array}{ccc} C_{rd}(\mathbb{T}, \mathbb{R}) & \longrightarrow & C_{rd}(\mathbb{T}, \mathbb{R}) \\ x & \mapsto & \widetilde{f}_{\mathbb{T}}[x] := t \to f(x(t)). \end{array}$$

The next definition gives the time scale embedding of the differential operator (3).

Definition 6 (*Time Scale Embedding of Differential Operators*)**.** The time scale embedding of the differential operator (3) is defined by

$$O_\Delta = \sum_{i=0}^{n} \widetilde{a}_{i\mathbb{T}} \cdot (\Delta^i \circ \widetilde{b}_{i\mathbb{T}}).$$

The two previous definitions are sufficient to define the time scale embedding of a given ordinary differential equation.

Definition 7 (*Time Scale Embedding of Differential Equations*)**.** The delta-differential embedding of an ordinary differential equation $O[x] = 0$, $x \in C^n([a, b], \mathbb{R})$, is given by $O_\Delta[x] = 0$, $x \in C^n_{rd}(\mathbb{T}^{\kappa^n}, \mathbb{R})$.

In order to define the delta-integral and the delta-variational embeddings (see Sections 7–9) we need to know how to embed an integral functional.

Definition 8 (*Time Scale Embedding of Integral Functionals*)**.** Let $L : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and $\mathcal{L}$ the functional defined by

$$\mathcal{L}(x) = \int_a^t L(s, x(s), D[x](s))ds = \int_a^t \widetilde{L}[x](s)ds.$$

The time scale embedding $\mathcal{L}_\Delta$ of $\mathcal{L}$ is given by

$$\mathcal{L}_\Delta(x) = \int_a^{\sigma(t)} L(s, x(s), \Delta[x](s))\Delta s = \int_a^{\sigma(t)} \widetilde{L}_{\mathbb{T}}[x](s)\Delta s.$$

5. Calculus of variations

The classical variational functional $\mathcal{L}$ is defined by

$$\mathcal{L}(x) = \int_a^b L(t, x(t), D[x](t))dt, \tag{4}$$

where $L : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth real-valued function called the Lagrangian (see, e.g., [8]). Functional (4) can be written, using the Lagrangian operator $\widetilde{L}$ (Definition 4), in the following equivalent form:

$$\mathcal{L}(x) = \int_a^b \widetilde{L}[x](t)dt.$$

The Euler–Lagrange equation associated to (4) is given (see, e.g., [8]) by

$$D[\tau \to \partial_3[L](\tau, x(\tau), D[x](\tau))](t) - \partial_2[L](t, x(t), D[x](t)) = 0, \tag{5}$$

$t \in [a, b]$, which we can write, equivalently, as

$$(D \circ \widetilde{\partial_3[L]})[x](t) - \widetilde{\partial_2[L]}[x](t) = 0.$$

Still another way to write the Euler–Lagrange equation consists in introducing the differential operator EL_L, called the *Euler–Lagrange operator*, given by

$$\mathrm{EL}_L := D \circ \widetilde{\partial_3[L]} - \widetilde{\partial_2[L]}.$$

We can then write the Euler–Lagrange equation simply as $\mathrm{EL}_L[x](t) = 0$, $t \in [a, b]$.

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6. Delta-differential embedding of the Euler–Lagrange equation

By Definition 6, the time scale delta embedding of the Euler–Lagrange operator EL_L gives the new operator

$$(\mathrm{EL}_L)_\Delta := \Delta \circ (\widetilde{\partial_3[L]})_{\mathbb{T}} - (\widetilde{\partial_2[L]})_{\mathbb{T}}.$$

As a consequence, we have the following lemma.

Lemma 9 (*Delta-Differential Embedding of the Euler–Lagrange Equation*). *The delta-differential embedding of the Euler–Lagrange equation is given by* $(\mathrm{EL}_L)_\Delta[x](t) = 0$, $t \in \mathbb{T}^{\kappa^2}$, *i.e.,*

$$(\Delta \circ \widetilde{\partial_3[L]}_{\mathbb{T}})[x](t) - \widetilde{\partial_2[L]}_{\mathbb{T}}[x](t) = 0 \tag{6}$$

for any $t \in \mathbb{T}^{\kappa^2}$.

In the discrete case $\mathbb{T} = [a, b] \cap h\mathbb{Z}$, we obtain from (6) the well-known discrete version of the Euler–Lagrange equation, often written as

$$\Delta_+ \circ \frac{\partial L}{\partial v}(t, x(t), \Delta_+ x(t)) - \frac{\partial L}{\partial x}(t, x(t), \Delta_+ x(t)) = 0, \tag{7}$$

$t \in \mathbb{T}^{\kappa^2}$, where $\Delta_+ f(t) = \frac{f(t+h)-f(t)}{h}$. The important point to note here is that, from the numerical point of view, Eq. (7) does not provide a good scheme. Let us see a simple example.

Example 4. Consider the Lagrangian $L(t, x, v) = \frac{1}{2}v^2 - U(x)$, where U is the potential energy and $(t, x, v) \in [a, b] \times \mathbb{R} \times \mathbb{R}$. Then the Euler–Lagrange equation (7) gives

$$\frac{x_{k+2} - 2x_{k+1} + x_k}{h^2} + \frac{\partial U}{\partial x}(x_k) = 0, \quad k = 0, \ldots, N-2, \tag{8}$$

where $N = \frac{b-a}{h}$ and $x_k = x(a + hk)$. This numerical scheme is of order 1, meaning that we make an error of order h at each step, which is of course not good.

In the next section, we show an alternative Euler–Lagrange equation to (7) that leads to more suitable numerical schemes. As we shall see in Section 9, this comes from the fact that the embedded Euler–Lagrange equation (6) is not coherent, meaning that it does not preserve the variational structure. As a consequence, the numerical scheme (8) is not *symplectic*, in contrast to the flow of the Lagrangian system (see [9]). In particular, the numerical scheme (8) dissipates energy artificially (see [10, Fig. 1, p. 364]).

7. Discrete variational embedding

Time scale embedding can be also used to define a delta analogue of the variational functional (4). Using Definition 8, and remembering that $\sigma(b) = b$, the time scale embedding of (4) is

$$\mathcal{L}_\Delta(x) = \int_a^b L(t, x(t), \Delta[x](t))\, \Delta t = \int_a^b \widetilde{L}_{\mathbb{T}}[x](t) \Delta t. \tag{9}$$

A calculus of variations on time scales for functionals of type (9) is developed in Section 9. Here, we just emphasize that, in the discrete case $\mathbb{T} = [a, b] \cap h\mathbb{Z}$, functional (9) reduces to the classical discrete Lagrangian functional

$$\mathcal{L}_\Delta(x) = h \sum_{k=0}^{N-1} L(t_k, x_k, \Delta_+ x_k), \tag{10}$$

where $N = \frac{b-a}{h}$, $x_k = x(a+hk)$ and $\Delta_+ x_k = \frac{x_{k+1}-x_k}{h}$, and that the Euler–Lagrange equation obtained by applying the discrete variational principle to (10) takes the form

$$\Delta_- \circ \frac{\partial L}{\partial v}(t, x(t), \Delta_+ x(t)) - \frac{\partial L}{\partial x}(t, x(t), \Delta_+ x(t)) = 0, \tag{11}$$

$t \in \mathbb{T}^\kappa_\kappa$, where Δ_- is the backward finite-difference operator defined by $\Delta_- f(t) = \frac{f(t)-f(t-h)}{h}$ [6,11].

The numerical scheme corresponding to the discrete variational embedding, i.e., to (11), is called in the literature a *variational integrator* [6,11]. The next example shows that the variational integrator associated with the problem in Example 4 is a better numerical scheme than (8).

Example 5. Consider the same Lagrangian as in Example 4 of Section 6: $L(t, x, v) = \frac{1}{2}v^2 - U(x)$, where $(t, x, v) \in [a, b] \times \mathbb{R} \times \mathbb{R}$. The Euler–Lagrange equation (11) can be written as

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + \frac{\partial U}{\partial x}(x_k) = 0, \quad k = 1, \dots, N-1,$$

where $N = \frac{b-a}{h}$ and $x_k = x(a + hk)$. This numerical scheme possess very good properties. In particular, it is easily seen that the order of approximation is now 2 and not of order 1, as in Example 4.

We remark that the form of $\mathcal{L}_\Delta$ given by (9) is not the usual one in the literature of time scales (see [12–14] and references therein). Indeed, in the literature of the calculus of variations on time scales, the following version of the Lagrangian functional is studied:

$$\mathcal{L}_{\mathbb{T}}^{\text{usual}}(x) = \int_a^b L(t, x(\sigma(t)), \Delta[x](t))\Delta t. \tag{12}$$

However, the composition of x with the forward jump σ found in (12) does not seem natural from the point of view of embedding.

8. Delta-integral embedding of the Euler–Lagrange equation in integral form

We begin by rewriting the classical Euler–Lagrange equation into integral form. Integrating (5), we obtain that

$$\partial_3[L](t, x(t), D[x](t)) = \int_a^t \partial_2[L](\tau, x(\tau), D[x](\tau))d\tau + c, \tag{13}$$

for some constant c and all $t \in [a, b]$ or, using the evaluation operator,

$$\widetilde{\partial_3[L]}[x](t) = \int_a^t \widetilde{\partial_2[L]}[x](\tau)d\tau + c.$$

Using Definition 8, we obtain the delta-integral embedding of the classical Euler–Lagrange equation (13).

Lemma 10 (*Delta-Integral Embedding of the Euler–Lagrange Equation in Integral Form*)**.** *The delta-integral embedding of the Euler–Lagrange equation (13) is given by*

$$\partial_3[L](t, x(t), \Delta[x](t)) = \int_a^{\sigma(t)} \partial_2[L](\tau, x(\tau), \Delta[x](\tau))\Delta\tau + c \tag{14}$$

or, equivalently, as

$$\widetilde{\partial_3[L]}_{\mathbb{T}}[x](t) = \int_a^{\sigma(t)} \widetilde{\partial_2[L]}_{\mathbb{T}}[x](\tau)\Delta\tau + c,$$

where c is a constant and $t \in \mathbb{T}^\kappa$.

Note that, in the particular case $\mathbb{T} = [a, b] \cap h\mathbb{Z}$, equation (14) gives the discrete Euler–Lagrange equation

$$\frac{\partial L}{\partial v}\left(t_k, x_k, \frac{x_{k+1} - x_k}{h}\right) = h\sum_{i=0}^{k} \frac{\partial L}{\partial x}\left(t_i, x_i, \frac{x_{i+1} - x_i}{h}\right) + c, \tag{15}$$

where $t_i = a + hi$, $i = 0, \dots, k$, $x_i = x(t_i)$, and $k = 0, \dots, N-1$. This numerical scheme is different from (7) and (11), and has not been discussed before in the literature with respect to embedding and coherence. This is done in Section 9.

9. The delta-variational embedding and coherence

Our next theorem shows that Eq. (14) can also be obtained from the least-action principle. In other words, Theorem 11 asserts that the delta-integral embedding of the classical Euler–Lagrange equation in integral form (13) and the delta-variational embedding are coherent.

Theorem 11. *If $\hat{x}$ is a local minimizer or maximizer to (9) subject to the boundary conditions $x(a) = x_a$ and $x(b) = x_b$, then $\hat{x}$ satisfies the Euler–Lagrange equation (14) for some constant c and all $t \in \mathbb{T}^\kappa$.*

Proof. Suppose that $\mathcal{L}_\Delta$ has a weak local extremum at $\hat{x}$. Let $x = \hat{x} + \varepsilon h$, where $\varepsilon \in \mathbb{R}$ is a small parameter, and $h \in C_{rd}^1$ such that $h(a) = h(b) = 0$. We consider

$$\phi(\varepsilon) := \mathcal{L}_\Delta(\hat{x} + \varepsilon h) = \int_a^b L(t, \hat{x}(t) + \varepsilon h(t), \Delta[\hat{x}](t) + \varepsilon\Delta[h](t))\Delta t.$$

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A necessary condition for $\hat{x}$ to be an extremizer is given by

$$\phi'(\varepsilon)|_{\varepsilon=0} = 0 \Leftrightarrow \int_a^b (\partial_2[L](t, \hat{x}(t), \Delta[\hat{x}](t))h(t) + \partial_3[L]\left(t, \hat{x}(t), \Delta[\hat{x}](t)\right)\Delta[h](t))\Delta t = 0. \tag{16}$$

The integration by parts formula (1) gives

$$\int_a^b \partial_2[L](t, \hat{x}(t), \Delta[\hat{x}](t))h(t)\Delta t = \int_a^t \partial_2[L](\tau, \hat{x}(\tau), \Delta[\hat{x}](\tau))\Delta\tau h(t)\Big|_{t=a}^{t=b} - \int_a^b \left(\int_a^{\sigma(t)} \partial_2[L](\tau, \hat{x}(\tau), \Delta[\hat{x}](\tau))\Delta\tau\, \Delta[h](t)\right)\Delta t.$$

Because $h(a) = h(b) = 0$, the necessary condition (16) can be written as

$$\int_a^b \left(\partial_3[L](t, \hat{x}(t), \Delta[\hat{x}](t)) - \int_a^{\sigma(t)} \partial_2[L](t, \hat{x}(t), \Delta[\hat{x}](t))\Delta\tau\right)\Delta[h](t)\,\Delta t = 0$$

for all $h \in C_{rd}^1$ such that $h(a) = h(b) = 0$. Thus, by the Dubois–Reymond Lemma (see [15, Lemma 4.1]), we have

$$\partial_3[L](t, \hat{x}(t), \Delta[\hat{x}](t)) = \int_a^{\sigma(t)} \partial_2[L](\tau, \hat{x}(\tau), \Delta[\hat{x}](\tau))\Delta\tau + c$$

for some $c \in \mathbb{R}$ and all $t \in \mathbb{T}^\kappa$. □

10. Conclusion

Given a variational functional and a corresponding Euler–Lagrange equation, the problem of coherence concerns the coincidence of a direct embedding of the given Euler–Lagrange equation with the one obtained from the application of the embedding to the variational functional followed by application of the least-action principle. An embedding is not always coherent, and a nontrivial problem is to find conditions under which the embedding can be made coherent. An example of this is given by the standard discrete embedding: the discrete embedding of the Euler–Lagrange equation gives (7) but the Euler–Lagrange equation (11) obtained by the standard discrete calculus of variations does not coincide. On the other hand, from the point of view of numerical integration of ordinary differential equations, we know that the discrete variational embedding is better than the direct discrete embedding of the Euler–Lagrange equation (see Example 5). The lack of coherence means that a pure algebraic discretization of the Euler–Lagrange equation is not good in general, because we miss some important dynamical properties of the equation which are encoded in the Lagrangian functional. A method to solve this default of coherence had been recently proposed in [6], and consists in rewriting the classical Euler–Lagrange equation (5) as an asymmetric differential equation using left and right derivatives. Inspired by the results of [16], here we propose a completely different point of view to embedding based on the Euler–Lagrange equation in integral form. For that we introduce a new delta-integral embedding (see Definition 8). Our main result shows that the delta-integral embedding and the delta-variational embedding are coherent for any possible discretization (Theorem 11 is valid on an arbitrary time scale).

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