A NON-DIFFERENTIABLE NOETHER'S THEOREM

by

Jacky Cresson[1,2] & Isabelle Greff[1,3]

1. Laboratoire de Mathématiques et de leur Appliquations de Pau, Université de Pau et des Pays de l'Adour, avenue de l'Université, BP 1155, 64013 Pau Cedex, France
2. Institut de Mécanique Céleste et de Calcul des Éphémérides, Observatoire de Paris, 77 avenue Denfert-Rochereau, 75014 Paris, France
3. Institut des Hautes Etudes Scientifiques, 35 routes de Chartres, 91440 Bures-sur-Yvette, France

Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie.

Emmy Noether, Invariante Variationsprobleme, 1918

Abstract. — In the framework of the non-differentiable embedding of Lagrangian systems, defined by Cresson and Greff in [**7**], we prove a Noether's theorem based on the lifting of one-parameter groups of diffeomorphisms.

Contents

2000 ***Mathematics Subject Classification.*** — 49S05, 49J05, 26B05, 70H03.
Key words and phrases. — Noether's theorem, Symmetries, First integrals, Non-differentiable calculus of variations, Lagrangian systems.

1. Introduction

This paper is a contribution to the idea of embedding of Lagrangian systems initiated in [**5**]. A review of the subject is given in [**4**]. An embedding of an ordinary or partial differential equation is a way to give a meaning to this equation over a larger set of solutions, like stochastic processes or non-differentiable functions. As an example, Schwartz's theory of distributions can be seen as an embedding theory. In this paper, we consider an extension of a particular class of differential equations which are Euler-Lagrange equations over non-differentiable functions described in [**7**]. The Euler-Lagrange equations are second order differential equations whose solutions correspond to critical points of a Lagrangian functional, [**2**]. Lagrangian systems cover a large set of dynamical behaviors and are widely used in classical mechanics. In [**13, 12**], Nottale introduce the idea that the space-time structure at the microscopic scale becomes non-differentiable. His goal is to recover the classical equations of quantum mechanics from those of classical mechanics. Using the fact that at the macroscopic scale the space-time is differentiable and the equations of mechanics are governed by a variational principle, called the least-action principle, he formulates a scale-relativity principle. Namely, the equations of motions over the non-differentiable space-time are given by the classical equation extended to non-differentiable solutions. This extension is done by choosing a different operator of differentiation on continuous functions. In [**7**], we defined the notion of non-differentiable embedding of differential equations and proved that the solutions of an embedded Euler-Lagrange equation correspond to critical points of a non-differentiable Lagrangian functional. In particular, the classical Newton's equation of Mechanics transforms into the Schrödinger equation by a non-differentiable embedding. This is summarized by the following diagram:

$$\begin{array}{ccc} \text{Lagrangian} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. Lagrangian} \\ \text{L.A.P.}\big\downarrow & & \big\downarrow\text{N.D.L.A.P.} \\ \text{Euler-Lagrange equation} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. Euler-Lagrange equation} \end{array}$$

where N.D. stands for non-differentiable, Emb. for embedding and L.A.P for least-action principle. Let us mention some other works on non-differentiable Euler-Lagrange equation in [**1, 3, 8**].

In this paper, we pursue our study of the non-differentiable embedding of Lagrangian systems. A classical result of Emmy Noether provides a relation between groups of symmetries of a given equation and constants of motion, i.e. first integrals. Precisely, if a Lagrangian system is invariant under a group of symmetries then it admits an explicit first integral. In the framework of the non-differentiable embedding of Lagrangian systems, we have then a natural question: Assume that the classical Lagrangian system is invariant under a group of symmetries, what can be said about the non-differentiable embedded Lagrangian system? In particular, do we have a non-differentiable notion of constants of motion? If yes, is it possible to extend the Noether's theorem? These questions can be summarized by the following diagram:

$$\begin{array}{ccc}
\text{invariance of Lagrangian} & \xrightarrow{\text{N.D. Emb}} & \text{invariance of N.D Lagrangian} \\
{\scriptstyle \text{Noether's thm.}}\big\downarrow & & \big\downarrow{\scriptstyle \text{N.D. Noether's thm.}} \\
\text{First integral} & \xrightarrow{\text{N.D. Emb}} & \text{N.D. First integral}
\end{array}$$

In this paper, we prove a non-differentiable Noether's theorem. Previous attempt in this direction has been made in [**6**] using a different formalism over non-differentiable functions and not in the context of the non-differentiable embedding of Lagrangian systems. In particular, the problem of the persistence of symmetries under embedding was not discussed.

The outline of the paper is as follows: first, we recall the framework of the non-differentiable calculus of variations introduced in [**7**]. In section 3, we remind classical results about group of symmetries, first integrals, and Noether's theorem. We then introduce the notion of invariance for a non-differentiable Lagrangian functional and discuss the problem of persistence of symmetries under a non-differentiable embedding. Section 4 is devoted to the proof of the non-differentiable Noether's theorem. We conclude with application to the Navier-Stokes equation.

2. Reminder about non-differentiable calculus of variations

We recall some notations and definitions from [**7**].

2.1. Definitions. — Let $d \in \mathbb{N}$ be a fixed integer, I an open set in $\mathbb{R}$, and $a, b \in \mathbb{R}$, $a < b$, such that $[a, b] \subset I$, be given in the whole paper. We denote by $\mathcal{F}(I, \mathbb{R}^d)$ the set of functions $x : I \to \mathbb{R}^d$.

Definition 1. — *(Hölderian functions) Let $x \in \mathcal{C}^0(I, \mathbb{R}^d)$. Let $t \in I$.*

1. *x is Hölder of Hölder exponent α, $0 < \alpha < 1$, at point t if*
$$\exists c > 0,\, \exists \eta > 0 \, s.t. \forall t' \in I \mid t - t' \mid \leq \eta \Rightarrow \|x(t) - x(t')\| \leq c \mid t - t' \mid^{\alpha},$$
where $\|\cdot\|$ is a norm on $\mathbb{R}^d$.
2. *x is α-Hölder and inverse Hölder with $0 < \alpha < 1$, at point t if*
$$\exists c, C \in \mathbb{R}^+,\, c < C,\, \exists \eta > 0 \, s.t. \forall t' \in I \mid t - t' \mid \leq \eta \Rightarrow$$
$$c \mid t - t' \mid^{\alpha} \leq \|x(t) - x(t')\| \leq C \mid t - t' \mid^{\alpha} .$$

A complex valued function is α-Hölder if its real and imaginary parts are α-Hölder. We denote by $H^{\alpha}(I, \mathbb{R}^d)$ the set of continuous functions α-Hölder. For explicit examples of α-Hölder and α-inverse Hölder functions we refer to ([**14**], p.168) in particular the Knopp or Takagi function.

2.2. The quantum derivative. — Let $x \in \mathcal{C}^0(I, \mathbb{R}^d)$. For any $\epsilon > 0$, the ϵ-scale derivative of x at point t is the quantity denoted by
$$\frac{\Box_\epsilon}{\Box t} : \mathcal{C}^0(I, \mathbb{R}^d) \to \mathcal{C}^0(I, \mathbb{C}^d)$$
and defined by
$$\frac{\Box_\epsilon x}{\Box t} := \frac{1}{2}\Big[\big(d_\epsilon^+ x + d_\epsilon^- x\big) + i\mu\big(d_\epsilon^+ x - d_\epsilon^- x\big)\Big],$$
where $\mu \in \{1, -1, 0, i, -i\}$ and
$$d_\epsilon^\sigma x(t) := \sigma \frac{x(t + \sigma\epsilon) - x(t)}{\epsilon},\ \sigma = \pm, \quad \forall t \in I.$$

Definition 2. — *Let $x \in \mathcal{C}^0(I, \mathbb{C}^d)$ be a continuous complex valued function. For all $\epsilon > 0$, the ϵ-scale derivative of x, denoted by $\frac{\Box_\epsilon x}{\Box t}$ is defined by*

$$\frac{\Box_\epsilon x}{\Box t} := \frac{\Box_\epsilon \mathrm{Re}(x)}{\Box t} + i\frac{\Box_\epsilon \mathrm{Im}(x)}{\Box t}, \tag{1}$$

where $\mathrm{Re}(x)$ *and* $\mathrm{Im}(x)$ *are the real and imaginary part of* x.

Let $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ be the subspace of $\mathcal{C}^0(I\times]0,1],\mathbb{R}^d)$ such that for any function $f\in\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ the limit $\lim\limits_{\epsilon\to 0} f(t,\epsilon)$ exists for any $t\in I$. We denote by E a complementary space of $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ in $\mathcal{C}^0(I\times]0,1],\mathbb{R}^d)$ and by π the projection onto $\mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)$ by

$$\begin{array}{rcl} \pi : \mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d)\oplus E & \to & \mathcal{C}^0_{conv}(I\times]0,1],\mathbb{R}^d) \\ f_{conv}+f_E & \mapsto & f_{conv}\,. \end{array}$$

We can then define the operator $\langle\,.\,\rangle$ by

$$\begin{array}{rcl} \langle\,.\,\rangle : \mathcal{C}^0(I\times]0,1],\mathbb{R}^d) & \to & \mathcal{F}(I,\mathbb{R}^d) \\ f & \mapsto & \langle f\rangle : t\mapsto \lim\limits_{\epsilon\to 0}\pi(f)(t,\epsilon)\,. \end{array}$$

Definition 3. — *Let us introduce the new operator* $\frac{\Box}{\Box t}$ *(without* ϵ*) on the space* $\mathcal{C}^0(I,\mathbb{R}^d)$ *by:*

$$\frac{\Box x}{\Box t} := \langle\frac{\Box_\epsilon x}{\Box t}\rangle. \tag{2}$$

The operator $\frac{\Box}{\Box t}$ acts on complex valued functions by $\mathbb{C}$-linearity.
For a differentiable function $x\in\mathcal{C}^1(I,\mathbb{R}^d)$, $\frac{\Box x}{\Box t}=\frac{dx}{dt}$, which is the classical derivative. More generally if $\frac{\Box^k}{\Box t^k}$ denotes $\frac{\Box^k}{\Box t^k}:=\frac{\Box}{\Box t}\circ\ldots\circ\frac{\Box}{\Box t}$ and $x\in\mathcal{C}^k(I,\mathbb{R}^d)$, $k\in\mathbb{N}$, then $\frac{\Box^k x}{\Box t^k}=\frac{d^k x}{dt^k}$.

Remark 1. — *This operator depends on the choice of the supplementary space* E*. However this dependence does not change the form of the formula we obtain in the following sections. We give an interpretation in* **[7]** *part I, section 4 of this dependency under a particular functional space and we then discuss the physical meaning of this dependence.*

In the following, we assume that a supplementary space E is fixed. We then derive the counter part of the classical Leibniz formula. The form of these formula does not depend on the choice made for the supplementary space E in definition 3.

The following lemma is an analogous of the standard Leibniz (product) rule for non-differentiable functions under the action of $\frac{\Box}{\Box t}$:

Lemma 1 **(□-Leibniz rule)**. — *Let $f \in H^{\alpha}(I, \mathbb{R}^d)$ and $g \in H^{\beta}(I, \mathbb{R}^d)$, with $\alpha + \beta > 1$,*

$$\frac{\Box}{\Box t}(f \cdot g) = \frac{\Box f}{\Box t} \cdot g + f \cdot \frac{\Box g}{\Box t}. \tag{3}$$

Let us note that for $\beta = \alpha$, we must have $\alpha > \frac{1}{2}$.

Proof. — Let us first consider real valued functions $f \in H^{\alpha}(I, \mathbb{R})$ and $g \in H^{\beta}(I, \mathbb{R})$, and start with d_{ϵ}^{+}, then

$$d_{\epsilon}^{+}(fg) = \left(d_{\epsilon}^{+} f\right) g + f \left(d_{\epsilon}^{+} g\right) + \epsilon \left(d_{\epsilon}^{+} f\right) \left(d_{\epsilon}^{+} g\right).$$

Since f and g are respectively α and β-Hölder, we have $|d_{\epsilon}^{+} f| \leq c_f \, \epsilon^{\alpha-1}$ and $|d_{\epsilon}^{+} g| \leq c_g \, \epsilon^{\beta-1}$, then $|\epsilon d_{\epsilon}^{+} f \, d_{\epsilon}^{+} g| \leq c_f \, c_g \, \epsilon^{\alpha+\beta-1}$. This quantity converge to 0, when ϵ goes to 0, since $\alpha + \beta > 1$, so that $\langle \epsilon d_{\epsilon}^{+}(f\, g) \rangle = 0$. The same holds for d_{ϵ}^{-}, we obtain

$$\langle d_{\epsilon}^{\sigma}(f\, g) \rangle = \langle d_{\epsilon}^{\sigma} f \rangle \, g + f \, \langle d_{\epsilon}^{\sigma} g \rangle.$$

By linearity, we get $\frac{\Box}{\Box t}(fg) = \frac{\Box f}{\Box t}.g + f.\frac{\Box g}{\Box t}$ for real valued functions. The generalization to vector valued functions $f \in H^{\alpha}(I, \mathbb{R}^d)$ and $g \in H^{\beta}(I, \mathbb{R}^d)$ is straightforward. □

Definition 4. — *We denote by $\mathcal{C}_{\Box}^{1}(I, \mathbb{R})$ the set of continuous functions $f \in \mathcal{C}^{0}(I, \mathbb{R})$ such that $\frac{\Box f}{\Box t} \in \mathcal{C}^{0}(I, \mathbb{R})$.*

Lemma 2. — *Let $f \in \mathcal{C}_{\Box}^{1}(I, \mathbb{R})$ such that*

$$\lim_{\epsilon \to 0} \int_a^b (\Box_{\epsilon} f)_E \, dt = 0 \tag{4}$$

then

$$\int_a^b \frac{\Box f(t)}{\Box t} \, dt = f(b) - f(a). \tag{5}$$

Proof. — With the definition of d_{ϵ}^{σ}, it is easy to check that $\lim_{\epsilon \to 0} \int_a^b d_{\epsilon}^{\sigma} f(t) \, dt = f(b) - f(a)$, $\sigma = \pm$. We deduce that

$$\lim_{\epsilon \to 0} \int_a^b \Box_{\epsilon} f(t) \, dt = f(b) - f(a).$$

Moreover,

$$\int_a^b \frac{\Box f(t)}{\Box t}\, dt = \lim_{\epsilon\to 0}\int_a^b \pi\left(\Box_\epsilon f(t)\right)\, dt.$$

As $\Box_\epsilon f(t) - \pi\left(\Box_\epsilon f(t)\right) = \left(\Box_\epsilon f(t)\right)_E$ and the condition (4), we deduce that

$$\lim_{\epsilon\to 0}\int_a^b \left(\Box_\epsilon f(t) - \pi\left[\Box_\epsilon f(t)\right]\right)\, dt = \lim_{\epsilon\to 0}\int_a^b \left(\Box_\epsilon f(t)\right)_E = 0.$$

We deduce the result. □

2.3. Non-differentiable calculus of variations. —

Definition 5. — *An admissible Lagrangian function* L *is a continuous function* $\mathrm{L} : \mathbb{R}\times\mathbb{R}^d\times\mathbb{C}^d \to \mathbb{C}$ *such that* $\mathrm{L}(t, x, v)$ *is holomorphic with respect to* v*, differentiable with respect to* x *and real when* $v \in \mathbb{R}$*.*

Let us consider an admissible Lagrangian $\mathrm{L} : \mathbb{R}\times\mathbb{R}^d\times\mathbb{C}^d \to \mathbb{C}$. A Lagrangian function defines a *functional* on $\mathcal{C}^1(I, \mathbb{R}^d)$, denoted by

$$\mathcal{L} : \mathcal{C}^1(I, \mathbb{R}^d) \to \mathbb{R}, \quad x \in \mathcal{C}^1(I, \mathbb{R}^d) \longmapsto \int_a^b \mathrm{L}\big(s, x(s), \frac{dx}{dt}(s)\big)\, ds\,. \tag{6}$$

The classical *calculus of variations* analyzes the behavior of $\mathcal{L}$ under small perturbations of the initial function x. The main ingredients are a notion of differentiable functional and extremal. Extremals of the functional $\mathcal{L}$ can be characterized by an ordinary differential equation of order 2, called the Euler-Lagrange equation (See for example **[2]** page 58).

Theorem 1. — *The extremals* $x \in \mathcal{C}^1(I, \mathbb{R}^d)$*of* $\mathcal{L}$ *coincide with the solutions of the Euler-Lagrange equation denoted by (EL) and defined by*

$$\frac{d}{dt}\left[\frac{\partial \mathrm{L}}{\partial v}\left(t, x(t), \frac{dx}{dt}(t)\right)\right] = \frac{\partial \mathrm{L}}{\partial x}\left(t, x(t), \frac{dx}{dt}(t)\right). \tag{EL}$$

The non-differentiable embedding procedure allows us to define a natural extension of the classical Euler-Lagrange equation in the non-differentiable context.

Definition 6. — *The non-differentiable Lagrangian functional* $\mathcal{L}_\Box$ *associated to* $\mathcal{L}$ *is given by*

$$\mathcal{L}_\Box : \mathcal{C}^1_\Box(I, \mathbb{R}^d) \to \mathbb{R}, \quad x \in \mathcal{C}^1_\Box(I, \mathbb{R}^d) \longmapsto \int_a^b \mathrm{L}\big(s, x(s), \frac{\Box x(s)}{\Box t}\big)\, ds\,. \tag{7}$$

Let $H_0^\beta := \{h \in H^\beta(I,\mathbb{R}^d),\ h(a)=h(b)=0\}$, and $x \in H^\alpha(I,\mathbb{R}^d)$ with $\alpha+\beta>1$. A H_0^β-variation of x is a function of the form $x+h$, with $h \in H_0^\beta$. For $x \in H^\alpha(I,\mathbb{R}^d)$ and $h \in H_0^\beta$, we denote by $D\mathcal{L}_\Box(x)(h)$ the quantity

$$\lim_{\epsilon\to 0} \frac{\mathcal{L}_\Box(x+\epsilon h)-\mathcal{L}_\Box(x)}{\epsilon}$$

if it exists and called the differential of $\mathcal{L}_\Box$ at point x in direction h. A H_0^β-extremal curve of the functional $\mathcal{L}_\Box$ is a curve $x \in H^\alpha(I,\mathbb{R}^d)$ satisfying

$$D\mathcal{L}_\Box(x)(h)=0, \quad \text{for any } h \in H_0^\beta.$$

Theorem 2 **(Non-differentiable least-action principle)**

Let $0<\alpha,\beta<1$ s.t. $\alpha+\beta>1$. Let L be an admissible Lagrangian function of class $\mathcal{C}^2$. We assume that $x \in H^\alpha(I,\mathbb{R}^d)$, such that $\frac{\Box\gamma}{\Box t} \in H^\alpha(I,\mathbb{R}^d)$ and $\frac{\partial L}{\partial v}(t,\gamma,\Box\gamma).h$ satisfies condition (4) for all $h \in H_0^\beta(I,\mathbb{R}^d)$. A curve $\gamma \in H^\alpha(I,\mathbb{R}^d)$ satisfying the following generalized Euler-Lagrange equation

$$\frac{\partial L}{\partial x}\big(t,\gamma(t),\frac{\Box\gamma(t)}{\Box t}\big)-\frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}(t,\gamma(t),\frac{\Box\gamma(t)}{\Box t})\right)=0\,. \qquad (NDEL)$$

is an extremal curve of the functional (7) on the space of variations $H_0^\beta(I,\mathbb{R}^d)$.

Proof. — The differential of $\mathcal{L}_\Box$ on $\gamma \in H^\alpha(I,\mathbb{R}^d)\cap\mathcal{C}^1_\Box(I,\mathbb{R}^d)$ is given by
(8)

$$D\mathcal{L}_\Box(\gamma)(h)=\int_a^b \left(\frac{\partial L}{\partial x}\Big(t,\gamma(t),\frac{\Box\gamma(t)}{\Box t}\Big)\cdot h(t)+\frac{\partial L}{\partial v}\big(t,\gamma(t),\frac{\Box\gamma(t)}{\Box t}\big)\cdot\frac{\Box h(t)}{\Box t}\right)dt.$$

for any $h \in H_0^\beta(I,\mathbb{R}^d)$.

As $\frac{\partial L}{\partial v}(t,\gamma(t),\frac{\Box\gamma(t)}{\Box t})$ is $H^\alpha(I,\mathbb{R}^d)$, and $h \in H_0^\beta(I,\mathbb{R}^d)$, with $\alpha+\beta>1$, we obtain using lemma 1

$$\int_a^b \frac{\partial L}{\partial v}\cdot\frac{\Box h(t)}{\Box t}\,dt=\int_a^b \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}\cdot h\right)dt-\int_a^b \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}\right)\cdot h\,.$$

As $\frac{\partial L}{\partial v}.h$ satisfies condition (4) for all $h \in H_0^\beta(I,\mathbb{R}^d)$, we obtain using lemma 2 and that $h(a)=h(b)=0$

$$\int_a^b \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}.h\right)dt=0.$$

The differential (8) becomes

(9)

$$D\mathcal{L}_\Box(\gamma)(h) = \int_a^b \left[\frac{\partial L}{\partial x}(t, \gamma(t), \frac{\Box\gamma(t)}{\Box t}) - \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial v}(t, \gamma(t), \frac{\Box\gamma(t)}{\Box t}) \right) \right] \cdot h(t)\, dt,$$

for all $h \in H_0^\beta$. □

3. Group of symmetries and invariance of functionals

3.1. Group of symmetries. — Symmetries are defined via the action of one parameter group of diffeomorphisms.

Definition 7. — *We call $\{\phi_s\}_{s\in\mathbb{R}}$ a one parameter group of diffeomorphisms $\phi_s : \mathbb{R}^d \to \mathbb{R}^d$, of class $\mathcal{C}^1$ satisfying*

i) $\phi_0 = \mathrm{Id}$,
ii) $\phi_s \circ \phi_u = \phi_{s+u}$.
iii) ϕ_s *is of class* $\mathcal{C}^1$ *with respect to* s.

Classical examples of symmetries are given by translations in a given direction u

$$\phi_s : x \mapsto x + su, \quad x \in \mathbb{R}^d$$

and rotations of angle θ

$$\phi_s : x \mapsto xe^{is\theta}, \quad x \in \mathbb{C}.$$

In [**6**] we use the related notion of *infinitesimal transformations*, instead of group of diffeomorphisms. They are obtained using a Taylor expansion of $y_t(s) = \phi_s(x(t))$ in a neighborhood of 0. We obtain

$$y_t(s) = y_t(0) + s \cdot \frac{dy_t}{ds}(0) + o(s).$$

As $\phi_0 = \mathrm{Id}$, we deduce that denoting by $\xi(t,x) = \dfrac{dy_t}{ds}(0)$ an *infinitesimal transformation* is of the form

$$x(t) \mapsto x(t) + s\xi(t, x(t)) + o(s).$$

3.2. Invariance of functionals and Noether's theorem. — In this section, we recall a classical result of E. Noether, [**11, 10**], which provides a relation between symmetries and first integrals, i.e. constants of motions. The classical notion of *first integral* for a dynamical systems can be defined in various ways leading to different generalized concepts of first integrals for non-differentiable dynamical system. We consider the following one:

Definition 8 **(First integral).** — *Let $J : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a function of class $\mathcal{C}^1$, then J is said to be a first integral of the ordinary differential equation $\dot{x}(t) = f(t, x(t))$, with $f \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ if for any solution x of the ordinary equation we have*

$$\frac{d}{dt}\big(J(t, x(t))\big) = 0 \quad \textit{for any } t \in \mathbb{R}.$$

The Euler-Lagrange equation is a second order differential equation. Therefore, a first integral for the Euler-Lagrange equation is a function $J : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that for any solution x of the Euler-Lagrange equation, we have

$$\frac{d}{dt}\big(J(t, x(t), \dot{x}(t))\big) = 0 \quad \text{for any } t \in \mathbb{R}.$$

Definition 9 **(Invariance).** — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms. An admissible Lagrangian L is said to be invariant under the action of Φ if it satisfies:*

$$L\Big(t, x(t), \frac{dx}{dt}(t)\Big) = L\Big(t, \phi_s(x(t)), \frac{d}{dt}\big(\phi_s(x(t))\big)\Big), \quad \forall s \in \mathbb{R}, \forall t \in \mathbb{R}, \tag{10}$$

for any solution x of the Euler-Lagrange equation.

A Lagrangian satisfying (10) will be called *classically invariant* under $\{\phi_s\}_{s\in\mathbb{R}}$, [**2**]. The Noether's theorem is based on the notion of invariance of Lagrangian under a group of symmetries. Let us recall the classical Noether's theorem, [**10**].

Theorem 3 **(Noether's theorem).** — *Let L be an admissible Lagrangian of class $\mathcal{C}^2$ invariant under $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$, a one parameter group of diffeomorphisms. Then, the function*

$$J : (t, x, v) \mapsto \frac{\partial L}{\partial v}(t, x, v) \cdot \frac{d\phi_s(x)}{ds} \mid_{s=0}$$

is a first integral of the Euler-Lagrange equation (EL).

3.3. The non-differentiable case. — The generalization of the notion of invariance of the Lagrangian to the non-differentiable case is quite natural and is deduced from the non-differentiable theory in [**7**]. This leads to the following definition.

Definition 10 **($\Box$-invariance)**. — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms. An admissible Lagrangian L is said to be $\Box$-invariant under the action of Φ if*

$$L(t, x(t), \frac{\Box x}{\Box t}(t)) = L(t, \phi_s(x(t)), \frac{\Box}{\Box t}(\phi_s(x(t)))), \quad \forall s \in \mathbb{R}, \quad \forall t \in I. \tag{11}$$

for any solution $x \in \mathcal{C}^1_\Box$ of the non-differentiable Euler-Lagrange equation (NDEL).

Remark 2. — *The regularity assumption on the family $\{\phi_s\}_{s\in\mathbb{R}}$ is related to the classical definition of invariance (10). In our case, we can weaken this assumption using for example family of homeomorphisms of class $\mathcal{C}^1_\Box$. However, as we have no examples of natural symmetries of this kind, we keep the classical definition.*

A natural question arising from the non-differentiable embedding theory of Lagrangian systems developed in [**7**] is the problem *persistence* of symmetries under embedding.

Problem 1 **(Persistence of invariance)**. — *Assume that a Lagrangian L is classically invariant under a group of symmetries $\{\phi_s\}_{s\in\mathbb{R}}$. Do we have the $\Box$-invariance of the Lagrangian L under $\{\phi_s\}_{s\in\mathbb{R}}$?*

This problem seems difficult. However, there exits one case where we can prove the persistence of invariance.

Definition 11 **(Strong invariance)**. — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms. An admissible Lagrangian L is said to be strongly invariant under the action of Φ if*

$$L(t, x, v) = L(t, \phi_s(x), \phi_s(v)), \quad \forall s \in \mathbb{R},\ \forall t \in I,\ \forall x \in \mathbb{R}^d,\ \forall v \in \mathbb{R}^d.$$

As an example we can consider the following Lagrangian L, given by:

$$L(t,x,v) = \frac{1}{2}\|v\|^2 - \frac{1}{\|x\|^2}.$$

If ϕ_s is a rotation, $\phi_s(x) := e^{is\theta}x$, then the Lagrangian L is strongly invariant.

***Definition 12* (□-commutation).** — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms, such that $\phi_s : \mathbb{C}^d \to \mathbb{C}^d$. Φ satisfies the □-commutation property, if*

$$\frac{\square}{\square t}(\phi_s(x(t))) = \phi_s\Big(\frac{\square x}{\square t}(t)\Big), \quad \forall s\in\mathbb{R}, \quad \forall t\in\mathbb{R}. \tag{12}$$

***Lemma 3* (Sufficient condition).** — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms, $\phi_s : \mathbb{C}^d \to \mathbb{C}^d$. If the Lagrangian L is strongly invariant and Φ satisfies the □-commutation property, then the Lagrangian L is □-invariant under the action of $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$.*

Proof. — Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ a one parameter group of diffeomorphisms. Let x be a solution of the non-differentiable Euler-Lagrange equation. Let $s\in\mathbb{R}$, applying definition 12 and condition (12), we obtain:

$$\begin{aligned} L\Big(t,\phi_s(x(t)),\frac{\square}{\square t}(\phi_s(x(t)))\Big) &= L\Big(t,\phi_s(x(t)),\phi_s(\frac{\square x}{\square t}(t))\Big), \\ &= L\Big(t,x(t),\frac{\square x}{\square t}(t)\Big), \end{aligned}$$

which concludes the proof. □

***Problem 2* (Commutation).** — *Let $\phi\in\mathcal{C}^1(\mathbb{C}^d,\mathbb{C}^d)$. Under which condition do we have the □-commutation*

$$\frac{\square}{\square t}\big(\phi(x)\big) = \phi\Big(\frac{\square}{\square t}(x)\Big)\ ?$$

Lemma 4. — *Let ϕ be a linear map, then ϕ satisfies the property of □-commutation.*

Proof. — As ϕ is linear on $\mathbb{C}^d$, there exists a matrix A such that $\phi : x \mapsto A\cdot x$. Hence, we have:

$$\frac{\square\phi(x)}{\square t} = \frac{\square(A\cdot x)}{\square t} = A\cdot\frac{\square x}{\square t} = \phi(\frac{\square x}{\square t}).$$

□

As a consequence, if L is strongly invariant under a linear group, then L is $\Box$-invariant.
We finish this section with a technical lemma which will be usefull in the proof of the non-differentiable Noether's theorem.

Lemma 5. — *Let $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$ be a one parameter group of diffeomorphisms $\phi_s : \mathbb{R}^d \to \mathbb{R}^d$, then we have*

$$\frac{d}{ds}\Big(\frac{\Box}{\Box t}(\phi_s(x(t)))\Big)|_{s=0} = \frac{\Box}{\Box t}\Big(\frac{d}{ds}(\phi_s(x(t)))|_{s=0}\Big). \tag{13}$$

Proof. — Using a Taylor expansion of $\phi_s(x(t))$ in $s = 0$, since $\phi_0(x(t)) = x(t)$, we obtain

$$\phi_s(x(t)) = x(t) + s\frac{d}{ds}(\phi_s(x(t)))|_{s=0} + s\,r(s, x(t)),$$

with $\lim\limits_{s\to 0} r(s,\cdot) = 0$. Then, since $\frac{\Box}{\Box t}$ is linear, we obtain

$$\frac{\Box\phi_s(x(t))}{\Box t} = \frac{\Box x(t)}{\Box t} + s\frac{\Box}{\Box t}\Big(\frac{d}{ds}(\phi_s(x(t)))|_{s=0}\Big) + s\frac{\Box r(s, x(t))}{\Box t}.$$

Taking the derivative with respect to s gives:

$$\frac{d}{ds}\Big(\frac{\Box\phi_s(x(t))}{\Box t}\Big) = \frac{\Box}{\Box t}\big(\frac{d}{ds}(\phi_s(x(t)))|_{s=0}\big) + \frac{\Box}{\Box t}\big(r(s, x(t))\big) + s\frac{d}{ds}\Big(\frac{\Box r(s, x(t))}{\Box t}\Big),$$

then, for $s = 0$, we deduce

$$\frac{d}{ds}\Big(\frac{\Box\phi_s(x(t))}{\Box t}\Big)|_{s=0} = \frac{\Box}{\Box t}\big(\frac{d}{ds}(\phi_s(x(t)))|_{s=0}\big) + \underbrace{\frac{\Box}{\Box t}\big(r(s, x(t))\big)|_{s=0}}_{=0}.$$

This concludes the proof. $\Box$

4. Non-differentiable Noether's theorem

As we have a notion of $\Box$-invariance of non-differentiable functionals, we can look for an analogous to Noether's theorem. This means that we need to define the corresponding notion of first integrals for $\Box$-differential equations. A generalization of definition 8 to non-differential curves comes from the non-differentiable theory of [**7**], and leads to the following definition.

Definition 13 **(Generalized first integral)**. — *A map $J : \mathbb{R} \times \mathbb{C}^d \to \mathbb{C}$ is a generalized first integral of an ordinary $\Box$-differentiable equation*

$$\frac{\Box x(t)}{\Box t} = f(t, x(t))$$

with $f \in \mathcal{C}^0(\mathbb{R} \times \mathbb{C}^d, \mathbb{C})$ if for any solution x

$$\frac{\Box}{\Box t}\big(J(t, x(t))\big) = 0 \quad \forall t \in \mathbb{R}.$$

A non-differentiable Euler-Lagrange equation is a second order $\Box$-differentiable equation, consequently an associated generalized first integral is a function $J : \mathbb{R} \times \mathbb{R}^d \times \mathbb{C}^d \to \mathbb{C}$ such that for any solution x of (NDEL), we have

$$\frac{\Box}{\Box t}\Big(J\big(t, x(t), \frac{\Box x(t)}{\Box t}\big)\Big) = 0 \qquad \forall t \in \mathbb{R}.$$

Theorem 4. — *Let L be a Lagrangian of class $\mathcal{C}^2$ $\Box$-invariant under $\Phi = \{\phi_s\}_{s\in\mathbb{R}}$, a one parameter group of diffeomorphisms, such that $\phi_s : \mathbb{C}^d \to \mathbb{C}^d$, for any $s \in \mathbb{R}$. Then, the function*

$$J : (t, x, v) \mapsto \frac{\partial L}{\partial v}(t, x, v) \cdot \frac{d\phi_s(x)}{ds}\,|_{s=0} \tag{14}$$

is a generalized first integral of the non-differentiable Euler-Lagrange equation (NDEL) on $H^\alpha(I, \mathbb{R}^d)$ with $\frac{1}{2} < \alpha < 1$.

Proof. — Let x be a solution of the non-differentiable Euler-Lagrange equation. Let $s \in \mathbb{R}$. As the Lagrangian is $\Box$-invariant under Φ,

$$L\Big(t, \phi_s(x(t)), \frac{\Box}{\Box t}\big(\phi_s(x(t))\big)\Big) = L\Big(t, x(t), \frac{\Box}{\Box t}x(t)\Big)\ ,\ \forall t \in I.$$

As a consequence, we obtain for any $s \in \mathbb{R}$

$$\frac{d}{ds}\left(L\Big(t, \phi_s(x(t)), \frac{\Box}{\Box t}\big(\phi_s(x(t))\big)\Big)\right) = 0. \tag{15}$$

On the other hand, we have for any $s \in \mathbb{R}$

$$\frac{d}{ds}\Big(L(t, \phi_s(x(t)), \frac{\Box}{\Box t}\big(\phi_s(x(t))\big)\Big) = \frac{\partial L}{\partial x}(\star_s(x)) \cdot \frac{d\phi_s(x(t))}{ds} + \frac{\partial L}{\partial v}(\star_s(x)) \cdot \frac{d}{ds}\Big(\frac{\Box \phi_s(x(t))}{\Box t}\Big),$$

where

$$\star_s(x) := \Big(t, \phi_s(x(t)), \frac{\Box}{\Box t}\big(\phi_s(x(t))\big)\Big).$$

Since (13) holds, we obtain for $s = 0$

$$\frac{d}{ds}\Big(L(\star_s(x))\Big)|_{s=0} = \frac{\partial L}{\partial x}(\star_0(x)) \cdot \frac{d\phi_s(x(t))}{ds}|_{s=0} + \frac{\partial L}{\partial v}(\star_0(x)) \cdot \frac{\Box}{\Box t}\left(\frac{d\phi_s(x(t))}{ds}\right)|_{s=0}.$$

Therefore, using (15) and since x is a solution of the non-differentiable Euler-Lagrange equation leads to

$$\frac{\square}{\square t}\left(\frac{\partial L}{\partial v}(\star_0(x))\right)\cdot\frac{d\phi_s(x(t))}{ds}\;|_{s=0}+\frac{\partial L}{\partial v}(\star_0(x))\cdot\frac{\square}{\square t}\left(\frac{d\phi_s(x(t))}{ds}\;|_{s=0}\right)=0.$$

As $x\in H^\alpha$, $\frac{\square x}{\square t}\in H^\alpha$ and $\frac{d}{ds}\phi_s$, $\frac{\partial L}{\partial v}$ continuous, with $2\alpha>1$, applying lemma 1 we obtain

$$\frac{\square}{\square t}\left(\frac{\partial L}{\partial v}(t,x(t),\frac{\square x}{\square t})\cdot\frac{d\phi_s(x(t))}{ds}\;|_{s=0}\right)=0.$$

This concludes the proof. □

5. Application

In [**7**] we define non-differentiable characteristics of a classical PDE. For the Navier-Stokes equation these non-differentiable characteristics coincide with critical points of a non-differentiable Lagrangian functional of the form

$$L(t,x,v)=\frac{1}{2}v^2-p(x,t), \tag{16}$$

where $x\in\mathbb{R}^d$, $v\in\mathbb{C}^d$ and $t\in\mathbb{R}$ over $H^{1/2}$. We refer to [**7**] for details.
Let $d=3$, we now study the Lagrangian (16) assuming that p is invariant with respect to the group of rotations around the vertical axis. With respect to our work on non-differentiable characteristics of the Navier-Stokes equation, this corresponds to consider the axisymmetric Navier-Stokes equations studied in [**9**]. Using the non-differentiable Noether's theorem we have the following result:

Theorem 5. — *Let L be the Lagrangian (16) where p is assumed invariant under the group of rotations $\Phi=\{\phi_\theta\}_{\theta\in\mathbb{R}}$ around the vertical axis given by*

$$\phi_\theta:\begin{array}{ccc}\mathbb{R}^3 & \longrightarrow & \mathbb{R}^3,\\ (x,y,z) & \longmapsto & (x\cos\theta-y\sin\theta,x\sin\theta+y\cos\theta,z).\end{array}$$

Then the function

$$J:\begin{array}{ccc}\mathbb{R}\times\mathbb{R}^3\times\mathbb{C}^3 & \longrightarrow & \mathbb{C}^3,\\ (t,(x,y,z),(v_x,v_y,v_z)) & \longmapsto & -yv_x+xv_y\end{array}$$

is a generalized first integral of the non-differentiable Euler-Lagrange equation

$$\frac{\square}{\square t}\left(\frac{\square x}{\square t}\right)=-\nabla_x p,$$

over H^α *with* $1/2 < \alpha < 1$.

Proof. — First, we extend Φ to $\mathbb{C}^3$ trivially. As p is invariant under the group Φ, and ϕ_θ is an isometry for each $\theta \in \mathbb{R}$, the Lagrangian L is strongly invariant under Φ. Moreover, as ϕ_θ is linear for each $\theta \in \mathbb{R}$, using lemma 4 we deduce that the group Φ satisfies $\Box$-commutation. Hence, applying lemma 3, we deduce that L is $\Box$-invariant under the action of Φ. We then apply theorem 4 to conclude. $\Box$

This result can be extended using the same argument on rotations, to the Lagrangian underlying the Schrödinger equation view as a non-differentiable Euler-Lagrange equation over $H^{1/2}$. Indeed, in this case, the function p is given by $1/\sqrt{x^2+y^2+z^2}$ defined on $\mathbb{R}^3 \setminus \{0\}$ and is invariant under each groups of rotations with respect to a fixed axis.

However, due to the limitation $1/2 < \alpha$, we cannot applied our result directly to give more informations on the non-differentiable characteristic of the Navier-Stokes equations or for the Schrödinger equation. The constraint on α is mainly due to the $\Box$-Leibniz rule.

References

[1] Ricardo Almeida and Delfim F. M. Torres. Generalized Euler-Lagrange equations for variational problems with scale derivatives, *Lett. Math. Phys.* 92(3), 221–229,2010.

[2] V. I. Arnold. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein.

[3] Zbigniew Bartosiewicz and Delfim F.M. Torres. Noether's theorem on time scales. *Journal of Mathematical Analysis and Applications*, 342(2):1220–1226, 2008

[4] Jacky Cresson. Introduction to embedding of lagrangian systems. *International journal for biomathematics and biostatistics*, to appear:12p, 2010.

[5] Jacky Cresson and Sébastien Darses. Stochastic embedding of dynamical systems. *J. Math. Phys.*, 48(7):072703, 54, 2007.

[6] Jacky Cresson, Gastão S. F. Frederico, and Delfim F. M. Torres. Constants of motion for non-differentiable quantum variational problems. *Topol. Methods Nonlinear Anal.*, 33(2):217–231, 2009.

[7] Jacky Cresson and Isabelle Greff. Non-differentiable embedding of lagrangian systems and partial differential equations. *Preprint Max-Plank-Institut für Mathematik in den Naturwissenschaften, Leipzig*, 16/2010:26p, 2010.

[8] Gastão S. F. Frederico, and Delfim F. M. Torres. A formulation of Noether's theorem for fractional problems of the calculus of variations. *Journal of Mathematics ans Applications*, 334(2):834-846, 2007.

[9] Isabelle Gallagher, Slim Ibrahim, and Mohamed Majdoub. Existence and uniqueness of solutions for an axisymmetric Navier-Stokes system. *Comm. Partial Differential Equations*, 26(5-6):883–907, 2001.

[10] Yvette Kosmann-Schwarzbach and Laurent Meersseman. *Les théorèmes de Noether Invariance et lois de conservation au XXe siècle*. Les éditions de l'Ecole Polytechnique. Ellipses, Paris, 2006.

[11] Emmy Noether. Invariante variationsprobleme. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, pages 235–257, 1918.

[12] Laurent Nottale. The scale-relativity program. *Chaos Solitons Fractals*, 10(2-3):459–468, 1999.

[13] Laurent Nottale. The theory of scale relativity. *Internat. J. Modern Phys. A*, 7(20):4899–4936, 1992.

[14] Claude Tricot. *Courbes et dimension fractale*. Springer-Verlag, Berlin, second edition, 1999. With a preface by Michel Mendès France.

Jacky Cresson[1,2]
Isabelle Greff[1,3]