Hamiltonian Chaos

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Abstract. We describe a new mechanism of instability for three degrees of freedom initially hyperbolic Hamiltonian systems. It is based on the specific geometry of simple resonances and the transversality-torsion phenomenon. It provides a solution to the gaps's problem occuring in the Arnold's mechanism of instability. We discuss the relevance of this work with respect to the Arnold's conjecture about topological instability of nearly integrable Hamiltonian systems.

Key-words. Hamiltonian systems, KAM theory, invariant manifolds, instability.

I INTRODUCTION

We consider a near-integrable Hamiltonian system with n degrees of freedom governed by the real analytic Hamiltonian

$$H(I,\theta,\epsilon)=h(I)+\epsilon f(I,\theta,\epsilon), \tag{1}$$

with the action-angle variables $(I,\theta)\in\mathbb{R}^n\times\mathbb{T}^n$, where $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$. For $\epsilon=0$, the system is integrable and the phase-space is foliated by n-dimensional (Lagrangians) tori. The K.A.M. (Kolmogorov-Arnold-Moser) theory states that under the non degeneracy of h ($\mid\bigtriangledown^2 h\mid\neq 0$), there remain many n-dimensional Lagrangian tori invariant under the flow which are slight deformation of the initial tori ($I=I_0$) located in a nonresonant area. For $n=2$, this result implies the stability of the action variables over infinite time. For $n\geq 3$, this does not prevent a drift of the orbit over a large part of the phase space in the perturbed system because the complement of the invariant tori is a *connected* set.

KAM theory refutes the ergodic hypothesis, according to which any neighborhood of Hamiltonians contains a Hamiltonian with ergodic flow on most of its energy surfaces. In the C^r topology with $r>2n+2$, this obiously not true.

For the dynamics in the complement of the KAM tori, we have the following conjecture of Arnold [2] : Let $n\geq 3$ and consider the space $C^\omega(\mathbb{R}^n,\mathbb{R})$ as the space of completely integrable Hamiltonians on the cotangent bundle $T^*\mathbb{T}^n=\mathbb{T}^n\times\mathbb{R}^n$ depending only on the action coordinates and endow these space with their natural C^k, $k\geq 2$ topology.

Asymptotic instability conjecture. - *There exists an open and dense set $\mathcal{D}$ in the space $C^\omega(\mathbb{R}^n,\mathbb{R})$ such that for every Hamiltonian $H\in\mathcal{D}$, and for every regular value h of$H$, for every pair O_1, O_2 of open sets in a connected component of $\mathcal{H}=H^{-1}(h)$ invariant under the flow defined by H, there exists a neighborhood $\mathcal{B}$ of H in $C^\omega(T^*(\mathbb{T}^n),\mathbb{R})$ and a residual set $\mathcal{R}$ in $\mathcal{B}$ such that for every $K\in\mathcal{R}$, there exists an orbit γ of the system defined by K which intersects O_1 and O_2.*

We stress that this conjecture is *asymptotic* which means that the topological instability which is called for, always occures if $\epsilon\neq 0$, despite the *metric* stability given by the K.A.M. theorem.

This conjecture follows a first example of Arnold [3], where he constructs a three degrees of freedom near-integrable Hamiltonian system which satisfies the assumptions of the K.A.M. theorem, but for which an instability of the action variables of order 1 occures. Arnold's example gives a first mechanism in order to prove his conjecture, based on *hyperbolic tori* along a simple resonance, connected by *heteroclinic* orbits, called a *chain*. Under a topological assumption on the dynamics near each torus of the chain, known as the *obstruction property*, he proved the existence of orbits of *drift* which connect the two extremal tori of the chain. We will call this mechanism, the *Arnold's mechanism*.

The existence of hyperbolic tori along simple resonance of near-integrable Hamiltonian systems has been proved by many authors and is known as *hyperbolic K.A.M. theory*. In order to apply Arnold's mechanism we must connect them via heteroclinic orbits. But, the hyperbolic tori that we construct are "too far" from each others with respect to there possible connecting neighborhood. This difficulty is known as the *gaps problem*. It already occures in *the initially hyperbolic* setting (also called a priori instable by Chierchia-Gallavotti [6]).

In this paper, we describe a new mechanism wich allows us to solve the gaps problem for a three degrees of freedom initially hyperbolic Arnold's model. We discuss the relevance of this work with respect to the Arnold's conjecture. All our methods are geometrical.
The idea is to construct a "skeleton" for drifting orbits based on Birkhoff unstable orbits and transversal homoclinic hyperbolic tori. A key result is that a transversal homoclinic hyperbolic torus with an ergodic flow generates *complete* hyperbolicity. We are then reduce to a problem of hyperbolic dynamics for which many tools already exist, the main one being the shadowing lemma ([1], [39]).

We refer to Xia [41], for a different approach based on Aubry-Mather sets and variational methods developed by J. Mather [30], and to the work of R. Douady [13] for a discussion of the Arnold's conjecture in the C^∞ class.

In the following, $f(x;y)$ (resp. $O_n(x;y)$) means a function of the argument x with the parameter y (resp. a function of the order of $\| x \|^n$ parametrized by y).

II HYPERBOLIC TORI AND TRANSITION CHAIN

We recall basic results about hyperbolic tori and transition chain [3].

A Hyperbolic tori

Let (M,Ω) denote a $2n$-dimensional smooth symplectic manifold and H a C^r function ($r\geq 2$). We denote by X_H the Hamiltonian vector field associated to H and defined by $i_{X_H}\Omega=-dH$ (where $i_{X_H}\Omega$ is the interior product of Ω by X_H), and by Φ the local flow of X_H. We denote by Φ_t the time-t diffeomorphism associated to Φ and ϕ the time-one map Φ_1.

Let f be a C^r , $r\geq 1$ diffeomorphism of a C^∞ manifold M which leaves invariant a compact C^r submanifold V of M. Given a Riemannian metric $\| \, \|$ on M, and a subbundle E of $T_V M$ invariant under Tf, we denote by $B_a(E)$ the unit ball in the fiber E_a, for $a\in V$, and write :

$$M(Tf_{|E})=\sup\{\| T_a f_{|E} \|, a\in V\},\ m(Tf_{|E})=\inf\{\sigma(T_a f_{|E}), a\in V\}, \tag{2}$$

where $\sigma(T_a f_{|E})=\inf\{\| T_a f(u) \|, u\in B_a(E)\}$, and $\| T_a f_{|E} \| = \sup\{\| T_a f(u) \|, u\in B_a(E)\}$.

Following [25], we define *hyperbolic tori* as :

Definition 1 *Let $k\in\mathbb{N}$, $k\geq 1$. An invariant torus $T\in M$ is said to be k-hyperbolic if there exists a continuous decomposition $T_T M=E^+ + E^0 + E^-$ and a Riemannian metric $\| \, \|$ on M such that :*
(i) $\dim E^+ = \dim E^- = k$,
(ii) the tangent bundle of T is contained in E^0,
(iii) $M(T\phi_{E^+})<1$, $m(T\phi_{E^-})>1$,
(iv) $M(T\phi_{E^0})=m(T\phi_{E_0})=1$.

Hyperbolic tori are usually obtain along 1-fold resonance surfaces [38]. We call such hyperbolic tori, KAM hyperbolic tori. In this case, there exists an analytic symplectic coordinate system $(I, \theta, s, u) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R} \times \mathbb{R}$ in a neighborhood of T in which the system takes the form :

$$H(I, \theta, s, u) = \omega I + \frac{1}{2} I^2 + \lambda s u + g(\theta, I, s, u), \tag{3}$$

where ω is a diophantine vector in $\mathbb{R}^2$, $0 < \lambda < 1$ and $g = O_3(I, su; I, s, u, \theta)$.

Assume that $\omega_n \neq 0$, then the submanifold $\mathcal{L}$ defined by $\theta_n = 0$ is transverse to the flow generated by the vector field X_H, in a neighborhood of T. There exists ([26], [7]) a coordinate system (θ, I, s, u) on $\mathcal{S}$ such that the first return map is given by

$$f(\theta, I, s, u) = (\theta + \nu + \omega I, I, \lambda s, \lambda^{-1} u) + r(\theta, I, s, u), \tag{4}$$

with $r = O_2(I, su; I, \theta, I, s, u)$, ν diophantine and $\omega \neq 0$ (definite torsion).

For non-KAM hyperbolic tori, the normal form (3) is no longer valid. Nevertheless, there exists [25] a symplectic coordinate system $(I, \theta, s, u) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R} \times \mathbb{R}$ such that the torus T is given by $T = \{I = 0, u = s = 0\}$, its invariant manifolds by $W^s(T) = \{I = 0, u = 0\}$ and $W^u(T) = \{I = 0, s = 0\}$ and the Hamiltonian takes the form

$$H(I, \theta, s, u) = C(\theta) I + I A(\theta) I + \lambda(\theta) s u + g(I, \theta, s, u), \tag{5}$$

where C, A and λ are of class C^r, $g = O_3(I, u, s; \theta)$ with $\partial_\theta g = 0$ on T, $\partial_s g = 0$ on $W^s(T)$ and $\partial_u g = 0$ on $W^u(T)$.

B Obstruction property and transition chains

Definition 2 *(Obstruction property) A hyperbolic torus possesses the obstruction property if all invariant submanifold V of $\mathcal{H}$ such that V and $W^s(T)$ intersect transversally, satisfy $W^u(T) \subset \overline{V}$.*

A hyperbolic torus satisfying the obstruction property is called a *transition* torus.

Definition 3 *A transition chain is a finite family $(T_i)_{i=1,\ldots,N}$ of transition tori such that $W^s(T_{i+1})$ and $W^u(T_i)$ intersect transversally for $i = 1, \ldots, N-1$.*

The main property of transition chain is [3] :

Theorem 1 *Let $(T_i)_{i=1,\ldots,N}$ be a transition chain, and U_1, U_N two arbitrary neighborhoods of T_1 and T_N. There exists an open set of orbits $\gamma(t)$ such that $\gamma(0) \in U_1$ and $\gamma(T) \in U_N$.*

Proof. - By definition 3, $W^u(T_1)$ and $W^s(T_2)$ intersect transversally. As T_2 is a transition torus, we deduce from definition 2, that $W^u(T_1) \subset \overline{W^u(T_2)}$. By induction, we obtain $W^u(T_1) \subset \overline{W^u(T_N)}$. Then, for each point $z \in W^u(T_1) \cap U_1$, the orbit of z under X_H, denoted $\gamma(z)$ intersect with U_N. $\square$

C Arnold's web

We denote $\omega(I) = \dfrac{\partial H_0}{\partial I}(I)$.

Definition 4 *Let $k \in \mathbb{Z}^n \setminus \{0\}$, the set of points $I \in \mathbb{R}^n$ such that $k.\omega(I) = 0$ is called the 1-fold resonance surface defined by k.*

We consider the *frequency map* defined by

$$\begin{array}{llll} \omega : \mathbb{R}^n & \to & \mathbb{RP}^{n-1}, \\ I & \mapsto & \omega(I) = (\omega_1(I), \ldots, \omega_n(I)). \end{array} \tag{6}$$

The set of all 1-fold resonance surfaces (called Arnold's web or stochastic web in the physical litterature), denoted $\mathcal{R}_1$, is the preimage of the rational hyperplan of $\mathbb{RP}^{n-1}$ by the frequency map.
If $n = 2$, resonances correspond to rational point on the projectiv line. The situation change if $n > 2$:

Lemma 1 *If $n > 2$, $\mathcal{R}_1$ is connected and dense on $\pi_I(\mathcal{H})$.*

Let I, $I' \in \pi_I(\mathcal{H})$, and $V(I)$, $V(I')$, two arbitrary neighborhoods of I and I' in $\pi_I(\mathcal{H})$, there exists a path γ in $\mathcal{R}_1$ such that $\gamma \cap V(I) \neq \emptyset$ and $\gamma \cap V(I') \neq \emptyset$.

Arnold's idea was to construct transition chain along $\mathcal{R}_1$ [3]. If such a construction is possible then the topological instability conjecture follows from theorem 1. But, as proved in section V, transition chains don't exist in general.

III MAIN RESULT

Let $0 < \epsilon$ and $0 < \mu << 1$, the 3 degrees of freedom initially hyperbolic Arnold's model is defined by

$$H_{\epsilon,\mu}(I, p, \theta, q) = \frac{1}{2}I^2 + \frac{1}{2}p^2 + \epsilon(\cos(q) - 1) + \epsilon\mu f(\theta, q), \tag{7}$$

where $I = (I_1, I_2) \in \mathbb{R}^2$, $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$, $p \in \mathbb{R}$, $q \in \mathbb{T}^1$ and $f(\theta, q)$ is an analytic function with Fourier expansion

$$f(\theta, q) = \sum_{k=(k_1,k_2)} f_k e^{i(k_1\theta + k_2 q)}. \tag{8}$$

Let $h \in \mathbb{R}^+$, $h > 0$ be a given energy and denote $\mathcal{H} = H^{-1}_{\epsilon,\mu}(h)$ the corresponding energy level. We denote by $\mathcal{R}$ the set defined by

$$\mathcal{R} = \{(I, p, \theta, q) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{T}^2 \times \mathbb{T}^1 \mid p = q = 0, (I, \theta) \in \mathcal{H}\}, \tag{9}$$

when $\mu = 0$.

The Hamiltonian (7) is a model for global dynamics near a simple resonance if $\mu = \epsilon^k$, with $k \in \mathbb{N}^*$ sufficiently large and $0 < \epsilon << 1$. In order to avoid a difficult *singular* perturbation problem, we assume μ independant of ϵ. A first step toward the Arnold's conjecture in this setting is then to prove :

Hyperbolic instability conjecture. - *We denote by π_I the canonical projection of $\mathbb{R}^n \times \mathbb{T}^n$ into $\mathbb{R}^n$. For a "generic" analytic perturbation f, there exists $\mu_0 > 0$ such that for $0 < \mu < \mu_0$ and each point I, $\tilde{I}$ in $\pi_I(\mathcal{R})$ and every neighborhood $V(I)$, $V(\tilde{I})$ of this two points, we can find a solution of the system $\gamma(t)$ such that $\gamma(0) \in V(I)$ and $\gamma(\tau) \in V(\tilde{I})$ with $\mid I - \tilde{I} \mid > \delta$, $\delta \in \mathbb{R}^*$ a constant independant of μ.*

In section IV, we prove that $\mathcal{R}$ is hyperbolic and foliated by hyperbolic tori (lemma 2). When $\mu \neq 0$, this set persists (it is denoted $\mathcal{R}_\mu$) and KAM-hyperbolic as well as non-KAM hyperbolic tori exists (lemma 3 and 10).

Our main result is :

Theorem 2 *We assume that f is such that :*

(i) Each non-KAM hyperbolic torus possesses a normal form given by (3) and ν in (4) is irrational.

(ii) Each hyperbolic torus is such that its stable and unstable manifold intersect transversally in $\mathcal{H}$.

(iii) A technical assumption about the dynamics in the complementary of hyperbolic tori on $\mathcal{R}_\mu$, detailed in equation (33).

Then, there exists for μ sufficiently small and each point I, $\tilde{I}$ in $\pi_I(\mathcal{R})$ and every neighborhood $V(I)$, $V(\tilde{I})$ of this two points, a solution of the system (7), $\gamma(t)$, such that $\gamma(0) \in V(I)$ and $\gamma(\tau) \in V(\tilde{I})$ with $\mid I - \tilde{I} \mid > \delta$, with $\delta \in \mathbb{R}_*^+$ a constant independant of μ.

The proof is given in section VII. All our methods and ideas extend to three degrees of freedom near-integrable Hamiltonian systems [11]. The method of proof follows ([1], [14], [15]).

We make some comments about our assumptions :

The first part of assumption (i) is here only to simplify some estimates in the proof of the theorem. We conjecture that our work remains valid with the weaker normal form (5).

The second part of (i) is fundamental and can't be avoid in our construction. As proved by Herman [19], one has generically a Liouville rotation numbers (see also [29]). We stress that it is important to work with *analytic* Hamiltonian systems in order to avoid Denjoy counter examples [40]. We refer to [18], [19] for complicated dynamics associated to C^∞ exact symplectic diffeomorphism of $T^*(\mathbb{T}^n)$.

We conjecture that assumption (ii) is generic for a suitable analytic topology (for example, the exponentially weighted norm [35]).

Assumption (iii) is more delicate. See comments below equation (33) for details.

A main property of our model Hamiltonian (7) is that it has definite torsion. It is fundamental in our approach since it implies $\omega \neq 0$ in (4). This assumption, coupled with (ii), gives rise to a (complete) hyperbolic dynamics in the neighborhood of each torus (which only posess a partial hyperbolic dynamics). We refer to [20] for dynamics connected with indefinite torsion.

IV GEOMETRY OF A SIMPLE RESONANCE

We refer to ([22], [39]) for basic results about hyperbolic sets.

For $\epsilon = \mu = 0$, the phase space is foliated by invariant tori of dimension 3.

For $\mu = 0$, we have a decoupled product of a pendulum $\frac{1}{2}p^2 + \epsilon(\cos(q) - 1)$ with 2 rotators $\frac{1}{2}I^2$.
For $0 \leq J \leq \sqrt{2h}$, we define

$$T_J = \{(I, p, \theta, q) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{T}^2 \times \mathbb{T}^1 \mid p = q = 0, I_1 = J, I_2 = \sqrt{2h - J}, \theta \in \mathbb{T}^2\}, \tag{10}$$

and $\omega(J) = (J, \sqrt{2h - J})$. We have [10] :

Lemma 2 *The set $\mathcal{R}$ is hyperbolic and foliated by 2 dimensional hyperbolic tori T_J, $0 \leq J \leq \sqrt{2h}$. Moreover, the dynamics on $\mathcal{R}$ is governed by the completly integrable Hamiltonian system $H_r(I, \theta) = \dfrac{1}{2}I^2$, $I \in \mathbb{R}^2$, $\theta \in \mathbb{T}^2$.*

Using the hyperbolic continuation theorem [39], we obtain [10]:

Lemma 3 *For $\mu \neq 0$, the set $\mathcal{R}$ persists. We denote $\mathcal{R}_\mu$ this set. The dynamics on $\mathcal{R}_\mu$ is conjugated to a 2 degrees of freedom near-integrable Hamiltonian system $H_{r,\mu}(I, \theta) = \frac{1}{2}I^2 + \mu g(I, \theta)$, $I \in \mathbb{R}^2$, $\theta \in \mathbb{T}^2$.*

We can precise the geometric structure of $\mathcal{R}_\mu$, using hyperbolic KAM theory. We denote

$$\Omega_{\gamma,\tau} = \left\{ \omega \in \mathbb{R}^2,\ \mid \omega . k \mid > \frac{\gamma}{\mid k \mid^\tau},\ \forall k \in \mathbb{Z}^2 \setminus \{0\} \right\}, \tag{11}$$

where $\mid k \mid = k_1 + k_2$ for $k = (k_1, k_2)$.

Lemma 4 *For $0 \leq J \leq \sqrt{2h}$ such that $\omega(J) \in \Omega_{\gamma,\tau}$, the hyperbolic torus T_J persists.*

Hyperbolic tori given by lemma 4 are called KAM-hyperbolic tori in the following.

V GAPS PROBLEM

We denote by $\mathcal{L}$ the Lebesgue measure and $\mathcal{G}$ the complement of the set of KAM hyperbolic tori $\mathcal{T}_{KAM}$ in $\mathcal{R}$. We have ([35], [33]) :

Lemma 5 $\mathcal{L}(\mathcal{G}) = O(\sqrt{\mu})$.

The set $\mathcal{G}$ is called the *gaps* set. In order to apply Arnold's mechanism, one usually wants to connect different tori in $\mathcal{T}_{KAM}$. Lemma 5 gives the greatest "gaps" between two tori of $\mathcal{T}_{KAM}$.

A Connecting neighborhood

Let $T \in \mathcal{T}_{KAM}$, there exists ([38], [17], [37]) a parametric representation of $W^s(T)$ and $W^u(T)$ given by, for $\rho > 0$ small enough :

$$\begin{aligned} W^s_\rho(T) &= \left\{(\theta, q, I^s(\theta,q), p^s(\theta,q), \theta \in \mathbb{T}^2, q \in]\rho, \pi+\rho[\right\}, \\ W^u_\rho(T) &= \left\{(\theta, q, I^u(\theta,q), p^u(\theta,q), \theta \in \mathbb{T}^2, q \in]-\pi-\rho, \rho[\right\}, \end{aligned} \tag{12}$$

for suitable functions $I^{s,u}$, $p^{s,u}$.

As usual, we define a section $\mathcal{S}$ of $W^s_\rho(T)$ and $W^u_\rho(T)$ by $q = \pi$. We denote by w^s (resp. w^u) the representation of $W^s(T)$ (resp. $W^u(T)$) in $\mathcal{S}$:

$$\begin{aligned} w^s &= \left\{(\theta, \pi, I^s(\theta,\pi), p^s(\theta,\pi)),\ \theta \in \mathbb{T}^2\right\}, \\ w^u &= \left\{(\theta, \pi, I^u(\theta,\pi), p^u(\theta,\pi)),\ \theta \in \mathbb{T}^2\right\}. \end{aligned} \tag{13}$$

The *splitting distance* is defined by

$$S(\theta) = I^u(\theta,\pi) - I^s(\theta,\pi),\ \theta \in \mathbb{T}^2. \tag{14}$$

Of course, if there exists $\theta_0 \in \mathbb{T}^2$ such that $S(\theta_0) = 0$ then, we have an intersection between $W^s(T)$ and $W^u(T)$ (this follows from the energy conservation).

We define the *transversality measure* at θ_0 by

$$T(\theta_0) = \mathrm{J}(S)(\theta_0), \tag{15}$$

where $\mathrm{J}(S)$ is the Jacobian matrix of S.

A Taylor's expension of (15) in μ gives

$$T(\theta_0) = \mu M(\theta_0) + O(\mu^2), \tag{16}$$

where $M(\theta_0)$ is called the *Melnikov*'s quantity.

We have the following lemma :

Lemma 6 *Let $\theta_0 \in \mathbb{T}^2$ such that $S(\theta_0) = 0$ and $M(\theta_0) \neq 0$, then for μ sufficiently small, and each $I' \in \mathbb{R}^2$ such that $T_{I'} \in \mathcal{T}_{KAM}$ and $\mid I - I' \mid \leq O(\mu^2)$, we have $W^u(T_I)$ and $W^s(T_{I'})$ which intersect transversally in $\mathcal{H}$.*

Proof. - We first recall a version of a quantitativ implicit function theorem : Let $(x,y) \mapsto f(x,y)$ be a function of class C^1, defined on $B^n_a \times B^m_b$, where B^n_a is a n dimensional ball around $x_0 \in \mathbb{R}^n$. Assume that $f(x_0,y_0) = 0$, and that the Jacobian matrix $T^{-1} = \dfrac{\partial f}{\partial x}(x_0,y_0)$ is invertible, and that

$$(i) \sup_{|y-y_0|\leq b} \mid f(x_0,y) \mid \leq \frac{1}{2} \parallel T \parallel^{-1} a, \quad (ii) \sup_{|x-x_0|\leq a,\ |y-y_0|\leq b} \parallel T^{-1} - \frac{\partial f}{\partial x} \parallel \leq \frac{1}{2} \parallel T \parallel^{-1}.$$

Then, there exists a unique C^1 function $x(y)$ such that $x(y_0) = x_0$ and $f(x(y), y) = 0$ for all $y \in B_b^m$.

By construction, we have

$$S(\theta; I, \mu) = S(\theta; I, 0) + \mu S_1(\theta; I, 0) + \sum_{k \geq 2} \mu^k S_k(\theta; I, 0), \tag{17}$$

with $S(\theta; I, 0) = 0$.

By assumptions, we have $S(\theta_0, I_0) = 0$ and $\dfrac{\partial S}{\partial \theta}(\theta_0, I_0) = k\mu$, for $k \in \mathbb{R}^*$ and μ sufficiently small.

We have

$$\sup_{|I-I_0| \leq b} \mid S(\theta_0, I) \mid \leq b. \sup_{|I-I_0| \leq b} \mid \frac{\partial S}{\partial I}(\theta_0, I) \mid . \tag{18}$$

As, the parametric representation are analytic with respect to I, there exists a constant $C_1 \in \mathbb{R}^+$ such that

$$\sup_{|I-I_0| \leq b} \mid \frac{\partial S}{\partial I}(\theta_0, I) \mid \leq C_1. \tag{19}$$

We deduce that

$$\sup_{|I-I_0| \leq b} \mid S(\theta_0, I) \mid \leq C_1 b. \tag{20}$$

A sufficient condition, in order to satisfy condition ii) of the quantitativ implicit function theorem is then

$$b \leq \frac{k\mu}{2C_1} a. \tag{21}$$

One can write condition ii) as

$$\sup_{|\theta-\theta_0| \leq a, \ |I-I_0| \leq b} \| \left(\frac{\partial S}{\partial \theta}(\theta_0, I_0) \right)^{-1} \| . \| \frac{\partial S}{\partial \theta}(\theta_0, I_0) - \frac{\partial S}{\partial \theta}(\theta, I) \| \leq \frac{1}{2}, \tag{22}$$

which is equivalent to

$$\| \frac{\partial S}{\partial \theta}(\theta_0, I_0) - \frac{\partial S}{\partial \theta}(\theta, I) \| \leq \frac{1}{2} k\mu. \tag{23}$$

We have

$$\frac{\partial S}{\partial \theta}(\theta_0, I_0) - \frac{\partial S}{\partial \theta}(\theta, I) = \frac{\partial S}{\partial \theta}(\theta_0, I_0) - \frac{\partial S}{\partial \theta}(\theta_0, I) + \frac{\partial S}{\partial \theta}(\theta_0, I) - \frac{\partial S}{\partial \theta}(\theta, I). \tag{24}$$

By analyticity of the parametrization of the stable and unstable manifolds, there exists $C_2 \in \mathbb{R}^+$, $C_3 \in \mathbb{R}^+$ such that

$$\sup_{|\theta-\theta_0| \leq a, \ |I-I_0| \leq b} \| \frac{\partial S}{\partial \theta}(\theta_0, I_0) - \frac{\partial S}{\partial \theta}(\theta, I) \| \leq aC_2 + bC_3. \tag{25}$$

A sufficient condition in order to have condition ii) is then

$$aC_2 + bC_3 \leq \frac{1}{2} k\mu. \tag{26}$$

Using (21), we obtain

$$a \leq \frac{k\mu}{4C_2}, \tag{27}$$

for μ sufficiently small.

We deduce, that

$$b \leq \frac{k^2\mu^2}{8C_2}. \tag{28}$$

This concludes the proof of the lemma. $\square$

Remark : we only nead parametrization of the stable and unstable manifolds of class C^2 in order to apply lemma 6.

Using lemma 6, it is not possible to construct transition chain of lenght independant of μ. Lemma 5 allows us to ensure the existence of hyperbolic KAM tori only if $\mid I - I' \mid = O(\sqrt{\mu})$. The connecting neighborhood of each hyperbolic KAM tori is of order $O(\mu^2)$ by lemma 6. Then, one can not connect, in general, hyperbolic KAM tori via transversal intersection of there stable and unstable manifold. This is the gaps problem.

B Short transition chains

We first introduce a definition.

Definition 5 *A finite family $(T_i)_{i=1,\dots,N}$ of transition torus such that T_1 (resp. T_N) is associated to the invariant curve C_{I_1} (resp. C_{I_N}). It is called a short transition chain, if for all $\mu > 0$ sufficiently small, we have $\mid I_1 - I_N \mid = O(\mu\sqrt{\mu})$.*

It is always possible to construct *short* transition chain in an initially hyperbolic Hamiltonian system [31]. We denote by $\mathcal{T}_I$ the diophantine hyperbolic torus associated to the invariant curve C_I with a diophantine rotation vector.

Lemma 7 *There exists an interval $[I_a, I_b]$ such that every two consecutive diophantine hyperbolic torus, denoted $\mathcal{T}_{I_1}$ and $\mathcal{T}_{I_2}$, are such that $\mid I_1 - I_2 \mid < k\mu^2$ for $k \in \mathbb{R}^+$.*

Proof. - We have $\mathcal{L}(\mathcal{G}) = O(\sqrt{\mu})$. Let $I_{a_i} = I_a + i\dfrac{I_b - I_a}{N}$, $i = 1, \dots, N$, and define N equal intervals $[I_{a_i}, I_{a_{i+1}}]$. Assume that $\mathcal{G}([I_{a_i}, I_{a_{i+1}}]) > k\mu^2$. As $\mathcal{L}(\mathcal{G}([I_a, I_b])) = O(\sqrt{\mu})$, we must have $N = O\left(\dfrac{1}{\mu\sqrt{\mu}}\right)$. Then, if $N = O\left(\dfrac{1}{\mu\sqrt{\mu}}\right)$, there exist $i \in \{1, \dots, N\}$ such that $\mathcal{L}(\mathcal{G}([I_{a_i}, I_{a_{i+1}}])) \leq k\mu^2$. $\square$

We then have

Lemma 8 *There exists short transition chain for $\mu > 0$ sufficiently small.*

Proof. - Let I_i, $i = 1, \dots, N$, be a finite family of KAM hyeprbolic tori with $I_i \in [I_a, I_b]$. For each tori T_{I_i}, we denote $C(I_i) = C_i\mu$ its connecting neighborhood. Let $C = \min_{i=1,\dots,N} C_i$, by lemma 7, there exists an interval $J \subset [a, b]$ such that two consecutive tori $I_i \in J$, $I_{i+1} \in J$, we have $\mid I_i - i_{i+1} \mid < C\mu^2$. Using lemma 6, this concludes the proof.$\square$

VI CONNECTORS AND HOMOCLINIC TUBE

We define two geometrical objects : connectors and homoclinic tube. Connectors are orbits in $\mathcal{R}_\mu$ which connect arbitrary small neighborhoods of two consecutive tori. A homoclinic tube is a neighborhood of a homoclinic orbit to a hyperbolic torus. Using connectors and homoclinic tube, we obtain a *skeleton* which allows us to construct drifting orbits in section VII.

A Connectors

1 Preliminaries

The dynamics on $\mathcal{R}$ is given by an integrable Hamiltonian system $\frac{1}{2}I^2$. We denote by $S_{\mathcal{R}}$ a global section of $\mathcal{R}$ defined by $\theta_2 = 0$. For $\mu \neq 0$, $S_{\mathcal{R}}$ is again a global section for $\mathcal{R}_\mu$ for μ sufficiently small.

Lemma 9 *The first return map is given by*

$$f_{\mathcal{R}}(\theta, I) = (\theta + \omega(I), I) + \mu r_f(\theta, I), \tag{29}$$

where $\theta \in \mathbb{T}$ and $I \in \mathbb{R}$ with $\partial_I(\omega) = 1$.

The proof follows from lemma 3.

Hyperbolic KAM and non-KAM tori are defined by

Lemma 10 *Let C be an invariant curve for $f_{\mathcal{R}}$ homotopic to 0, with an irrational rotation number, then C is associated to a hyperbolic torus for μ sufficiently small.*

We refer to ([25], [34]) for a proof.

For $\mu = 0$, we denote by C_{I_0} the invariant curve associated to $I = I_0$ in $S_{\mathcal{R}}$. Let C be an invariant curve for $\mu \neq 0$ homotopic to 0, and D_C the domain bounded by C. We then have a natural order on invariant curves.

Definition 6 *Let C_1 and C_2 be two invariant curves for $f_{\mathcal{R}}$, homotopic to 0. We say that $C_1 < C_2$ if $C_1 \subset D_{C_2}$.*

This order on invariant curves induce the same notion for hyperbolic tori.

Definition 7 *Let T_1 and T_2 be two hyperbolic tori associated to invariant curves C_1 and C_2 for $f_{\mathcal{R}}$. We say that $T_1 < T_2$ if $C_1 < C_2$.*

We say that T_1 and T_2 are consecutive hyperbolic tori if there are no other hyperbolic tori between T_1 and T_2 for the previous order.

2 Reminder about Birkhoff regions of instability

By lemma 9, the dynamics on $\mathcal{R}_\mu$ is given by a perturbed twist diffeomorphism. We recall classical results about twist diffeomorphisms, mainly due to Birkhoff [4], Mather [28] and Le Calvez [24].

We denote $\mathbf{A} = \mathbb{T} \times \mathbb{R}$. We consider area preserving diffeomorphism of $\mathbf{A}$, of class C^∞, in a neighborhood of an elliptic fixed point. This diffeomorphism possess in a neighborhood of this fixed point a set of invariants graphs ordered by there rotation numbers. This set is not empty by the KAM theorem.

Definition 8 *A region of $\mathbf{A}$, bounded by two invariant graphs, which contain no other invariant graphs, is called a Birkhoff region of instability.*

The existence of such regions is due to Birkhoff [4].

We denote by C a region bounded by two invariant graphs C^+ and C^-.

Proposition 1 *If V^+ and V^- are two neighborhoods of C^+ and C^- respectively, there exists $x \in V^+$ and $n \in \mathbb{N}$ such that $f^n(x) \in V^+$.*

By applying this proposition iteratively, we obtain ([28], [24]) :

Theorem 3 *There exists $x \in C$ such that*

$$\lim_{n\to\infty} d(f^n(x), C^-) = \lim_{n\to\infty} d(f^n(x), C^+) = 0. \tag{30}$$

For all neighborhood V of C^+, there exists $x \in V$ such that $\lim_{n\to\infty} d(f^n(x), C^-) = 0$.

One can prove also the existence of other structures (Aubry-Mather sets).

3 Connectors and a technical assumption

Let T_1 and T_2 be two consecutive tori and C_1 and C_2 there associated invariant curves. By definition 8, C_1 and C_2 bound a Birkhoff region of instability in $S_{\mathcal{R}}$. We then deduce from theorem 3 and lemma 9 :

Lemma 11 *Let T_1 and T_2 be two consecutive tori. We denote by V_1 and V_2 two neighborhoods of T_1 and T_2. There exists an orbit γ of the system such that $\gamma(0) \in V_1$ and $\gamma(\tau) \in V_2$.*

We call this orbit a *connector*. We have $\gamma \subset \mathcal{R}_\mu$. If γ is isolated in $S_{\mathcal{R}}$, then, as $\mathcal{R}_\mu$ is hyperbolic, we deduce that γ is an isolated invariant set in $\mathcal{H}$.

For each T_i, $i \in \mathbb{N}$, we denote

$$S^-_{I_i} = \{(\theta, s, I, u) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \ \mid s \mid < \lambda^{\tilde{n}_{I_i}}, \mid u \mid < \lambda^{\tilde{n}_{I_i}}, I = I_0 - \delta\}, \tag{31}$$

with $\lambda^{\tilde{n}_{I_i}} > \delta > \lambda^{2\tilde{n}_{I_i}}$, and $\tilde{n}_{I_i} \in \mathbb{N}$,

$$S^+_{I_i} = \{(\theta, s, I, u) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \ \mid s \mid < \lambda^{\tilde{n}_{I_i}}, \mid u \mid < \lambda^{\tilde{n}_{I_i}}, I = I_0 + \delta\}. \tag{32}$$

Let T_{I_i} and $T_{I_{i+1}}$ be two consecutive hyperbolic tori and $\gamma_{i,i+1}$ the connecting orbit between T_{I_i} and $T_{I_{i+1}}$. We denote by q_i^+ (resp. q_{i+1}^-) the intersection of $\gamma_{i,i+1}$ with $S^+_{I_i}$ (resp. $S^-_{I_{i+1}}$).

We make the following technical assumption :

(iii) For each $i \in \mathbb{N}$, μ sufficiently small and $\tilde{n}_{I_i}$ sufficiently great, the connexion map is given by

$$\begin{array}{rccl} C: & S^+_{I_i} & \to & S^-_{I_{i+1}}, \\ & x & \mapsto & q^-_{i+1} + C.x + r_c(x;\mu), \end{array} \tag{33}$$

where $x = z - q_i^+$, $r_c(x;0) = 0$ and $r_c(x;\mu) = O_2(I, su)$, and

$$C = Id + \mu M_i, \tag{34}$$

where M_i is an arbitrary matrix.

This assumption is weak, as it only provides a constraint on C when $\mu \to 0$. In fact, we can assume that $C \to C_0$, with C_0 such that $C_0 \circ F_n \circ \Pi \circ F_{n'}$ is hyperbolic for n and n' sufficiently great, with Π as in (37) and F_n as in (45). By lemma 15, $F_n \circ \Pi \circ F_{n'}$ is hyperbolic. As the set of hyperbolic isomorphism is open and dense in $GL(\mathbb{R}^n)$, we conjecture that (34) with C_0 instead of Id, is generic.

B Homoclinic tube

Let T_{I_i} be a hyperbolic torus such that :

i) $W^s(T_{I_i}) = W^u(T_{I_i})$ and $W^s(T_{I_i}) \subset T_{I_i}$ if $\mu = 0$.

ii) $W^s(T_{I_i})$ and $W^u(T_{I_i})$ intersect transversally when $\mu \neq 0$ with an angle σ of order μ.

Let γ be a homoclinic orbit. We denote $z = (\theta, s, I, u) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The homoclinic orbit intersects the section S in $p^+ \in W^s(T_{I_i})$ and $p^- \in W^u(T_{I_i})$. We have $p^+ = (\theta^+, s^+, 0, 0)$ and $p^- = (\theta^-, 0, 0, u^-)$. We define two neighborhoods of p^+ and p^- such that

$$\begin{array}{rcl} B^+ & = & \{\mid \theta - \theta^+ \mid \leq \mu, \mid s - s^+ \mid \leq \mu^2, \mid I \mid \leq \mu^2, \mid u \mid \leq \mu^2\}, \\ B^- & = & \{\mid \theta - \theta^- \mid \leq \mu, \mid s \mid \leq \mu^2, \mid I \mid \leq \mu^2, \mid u - u^- \mid \leq \mu^2\}. \end{array} \tag{35}$$

We denote by h the linearized map of Φ along γ. The map $h : B^- \to B^+$ satisfies the following conditions :

(a) $h(p^-) = p^+$,
(b) For $p = (\theta, 0, I_0, u) \in B^-$, we have $Th(T_p W^u(T)) = T_{h(p)} W^s(T)$ for $\mu = 0$.

(c) For $p=(\theta,s,0,u)$, we have $h(p)=(\theta',s',0,u')$ for $\mu=0$,
(d) $Th(T_pW^u(T))$ and $T_{h(p)}W^s(T)$ are transversal for $\mu\neq 0$.

Condition (b) corresponds to the coincidence of $W^s(T)$ and $W^u(T)$ when $\mu=0$. Condition (c) comes from the fact that I is a first integral when $\mu=0$. Condition (d) is related to the transversality of the intersection between the stable and unstable manifold.

We have the following lemma :

Lemma 12 *The map $h: B^- \to B^+$ defined by*

$$h(x)=p^+ + \Pi.x + r_h(x;\mu), \tag{36}$$

where $x=z-p^-$, and Π is a matrix of the form

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ \sigma & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \tag{37}$$

in the basis $(e_\theta, e_s, e_I, e_u)$, with $a,b,c,d\in\mathbb{R}$ such that $ad-bc=1$, $\sigma\neq 0$ and $r_h(x,\mu)=O_2(I,su)$, $r_h(x;0)=0$, satisfies (a),(b),(c) and (d).

We conjecture that the form (37) of Π is generic, up to a change of coordinate system. For our purpose, the following lemma is sufficient :

Lemma 13 *Let T be a hyperbolic torus of the Arnold's model such that $W^s(T)$ and $W^u(T)$ intersect transversally when $\mu\neq 0$. Then, the homoclinic map h is defined as in lemma 12.*

We refer to [10] for a proof.

VII SKETCH OF PROOF OF THE THEOREM

Let $T_1,\dots,T_N$ be a finite family of hyperbolic torus. We are going to prove that there exists an open set of orbits along the previous family of hyperbolic tori. The proof is done by induction on the number of tori.

We define the following subsets of $S^-_{I_i}$ and $S^+_{I_i}$, $i=1,\dots,N$:

$$\begin{array}{rcl} E^-_{I_i} & = & \{(\theta,s,I,u)\in\mathbb{T}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\ I=I_i-\delta, \mid s\mid < \lambda^{n_i^-}, \mid u-u^-\lambda^{n_i^-}\mid < \lambda^{2n_i^-}\}, \\ E^+_{I_i} & = & \{(\theta,s,I,u)\in\mathbb{T}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\ I=I_i+\delta, \mid s\mid < \lambda^{n_i^+}, \mid u-u^-\lambda^{n_i^+}\mid < \lambda^{2n_i^+}\}, \end{array} \tag{38}$$

The idea is the following : choose n_i^-, n_i^+ such that the map defined by $\Psi=\Psi_N\circ\ldots\circ\Psi_1$ from $E^-_{I_1}$ to $E^+_{I_N}$, where $\Psi_i = C\circ f^{n_i^+}\circ h\circ f^{n_i^-}$, is defined on an open set and is hyperbolic. Then, via the classical shadowing lemma, we deduce the existence of orbits which connect a neighborhood of T_1 and T_N.
Hyperbolicity follows from the transversality-torsion phenomenon (see [9]). The rest follows by construction.

We denote

$$\tilde{n} = \max_{i=1,\dots,N} \tilde{n}_{I_i}. \tag{39}$$

It is always possible to choose

$$n_i^+ = \tilde{n}, \tag{40}$$

in our construction. The main problem is to precise the value of n_i^- recursively.

We have

Theorem 4 *If $n_i^- = (i-1)\tilde{n} + n_0^-$, with n_0^- sufficiently great, then there exists an open set $V \subset E_{I_1}^-$ such that $\Psi(V) \subset E_{I_N}^+$ and Ψ is hyperbolic.*

This theorem is not completely constructive as n_0^- is not precised. We give a lower bound for the choice of n_0^-.

We first recall the property of ergodization for irrational rotation flow : Let $R_\nu : \theta \to \theta + \nu$ be a linear rotation of the circle of frequency ν. It is well-known that when ν is irrational, the orbits of R_ν are transitive (they fill all the circle densely). Let $\theta_0 \in \mathbb{S}^1$, and ν irrational, then following S. Dumas [12], the denseness of $\bigcup_{n\in\mathbb{N}} (\theta_0 + n\nu)$ implies an "ergodisation" result; i-e. given $R > 0$, there exists a $N > 0$, such that no point in $\mathbb{S}^1$ is more than R-distant from any orbit $\bigcup_{0\leq n\leq N} (\theta_0 + n\nu)$. The property of ergodizing the torus to within R after time N is independant of initial condition.

Let N_i be the ergodization time to within μ for each torus T_i, $i = 1, \dots, N$. Then, we must take n_0^- such that

$$n_0^- > \max_{1\leq i\leq N} N_i. \tag{41}$$

This number can be very big, as we have Liouville rotation numbers (if ν is diophantine (11), then the ergodization time is given by $\gamma R^{-\tau}$ and is optimal [5]).

Of course theorem 4 implies our main result. We first prove several technical lemmas.

Lemma 14 *For $z \in E_{I_i}^-$, we have $f^{n_i^-}(z) \in B_{I_i}^-$.*

Proof. - We first look for $f_l^n(z)$ where $f_l(z) = (\theta + \nu + \omega I, \lambda s, I, \lambda^{-1}u)$. We have $f_l^n(z) = (\theta + n\nu + n\omega I, \lambda^n s, I, \lambda^{-n}u)$. we denote $\tilde{z} = f_l^n(z)$. By definition of $E_{I_i}^-$, we obtain

$$\tilde{\theta} = \theta + n\nu + n\omega I_i + n\omega\delta,\ \ \tilde{s} = \lambda^{2n},\ \ \tilde{I} = I_i - \delta,\ \ |\ \tilde{u} - u^-\ | < \lambda^n. \tag{42}$$

As $r(\theta, s, I, u) = O_2(I, su)$, we deduce that

$$\|\ r\ \| \leq \lambda^{2n}. \tag{43}$$

As ν is irrational, for n_i^- sufficiently great, we have $|\ \tilde{\theta} - \theta^-\ | < \dfrac{\mu}{2}$ (*). From (*) and (42), we deduce that $|\ \theta - \theta^-\ | < \mu$. Then, using (**), (42) and (43), we obtain $f^{n_i^-}(z) \in B_{I_i}^-$. $\square$

We denote

$$f^n(z) = (n\nu, 0, 0, 0) + F_n.z + r_n(z), \tag{44}$$

where

$$F_n = \begin{pmatrix} 1 & 0 & n\omega & 0 \\ 0 & \lambda^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda^{-n} \end{pmatrix}. \tag{45}$$

We have

Lemma 15 *Let $g : E_{I_i}^- \to B_{I_i}^+$ defined by $g = h \circ f^n$, we write*

$$g(x) = p^+ + G.x + r_g(x;\mu), \tag{46}$$

with $r_g(x;\mu) = O_2(I, su)$ and

$$G = \begin{pmatrix} 1 & 0 & n\omega & 0 \\ 0 & a\lambda^n & 0 & b\lambda^{-n} \\ \sigma & 0 & \sigma n\omega + 1 & 0 \\ 0 & c\lambda^n & 0 & d\lambda^{-n} \end{pmatrix}. \tag{47}$$

The map g is hyperbolic.

Proof. - The matrix G is defined by $G = \Pi \circ F_n$. The caracteristic polynomial of G is given by

$$P(x) = (x^2 - x(2n\omega + 2) + 1)(x^2 - xa(n) + 1), \tag{48}$$

with $a(n) = a\lambda^n + d\lambda^{-n}$.

We first study the second factor of this polynomial. The eigenvalues are given by

$$x_\pm = \frac{a(n)}{2}(1 \pm \sqrt{1 - \frac{4}{a(n)^2}}). \tag{49}$$

For n sufficiently great, we have

$$x_\pm = \frac{d\lambda^{-n}}{2}(1 \pm (1 + \frac{2}{a(n)^2})), \tag{50}$$

so that we have hyperbolic directions, whose Lyapounov exponents are (up to negligeable factors), $x_+ \sim d\lambda^{-n}$ and $x_- \sim d^{-1}\lambda^n$.

For the first factor, the eigenvalues are given by

$$x_\pm = 1 + \frac{\sigma n\omega}{2} \pm \frac{1}{2}(\sigma^2 (n\omega)^2 + 4\sigma n\omega)^{\frac{1}{2}}. \tag{51}$$

As $\sigma \neq 0$ and $\omega \neq 0$ then, $x_\pm \neq \pm 1$. This concludes the proof of the lemma. □

We see that for $\sigma = 0$ we have $x_\pm = 1$. Moreover, if $\omega = 0$, we have also $x_\pm = 1$. The constant ω comes from the torsion of the flow on the torus, and σ comes from the transversality of the intersection betwwen the stable and unstable manifold of the torus. This justify the terminology of transversality-torsion phenomenon.

We make the following conjecture : Let T be a hyperbolic torus whose flow is ergodic with non degenerate torsion, and its stable and unstable manifold intersect transversally, then there exists a hyperbolic invariant set B near T, and the dynamics on B is conjugated to a Bernouilli shift.

We define the following subset of $B^+_{I_i}$:

$$\tilde{B}^+_{I_i} = \{(\theta, s, I, u) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mid u - u^- \lambda^{n_i^+} \mid < \lambda^{2n_i^+}\}. \tag{52}$$

Lemma 16 *If* $n_i^+ > \frac{n_i^-}{2} + \frac{1}{2}\frac{\log 2}{\log \lambda^{-1}}$, *then for* $z \in B^+_{I_i}$, *we have* $f^{n_i^+}(z) \in E^+_{I_i}$.

Then, we have the following lemma :

Lemma 17 *Let* $n^-_{i+1} = n_i^- + \tilde{n}$, *then* $\Psi_i : E^-_{I_i} \to E^+_{I_{i+1}}$ *is hyperbolic and there exists an open set* $V_i \subset E^-_{I_i}$ *such that* $\Psi_i(V_i) \subset E^+_{I_{i+1}}$.

The size of the drift set decreases exponentially fast with the number of tori. Moreover, if $\tilde{n} = \frac{1}{\log \lambda} \log\left(\frac{1}{\mu}\right)$, then it decreases also exponentially fast in μ.

Remark. - The time of instability n_u is then given by $n_u = \frac{N(N+3)}{2}\tilde{n} + Nn_0^-$. If all the tori are KAM-hyperbolic, then one can choose $n_0^- = \tilde{n}$. In the initially hyperbolic Arnold's example, we have $\tilde{n} = \frac{1}{\log \lambda} \log\left(\frac{1}{\mu}\right)$. Then, the time of instability is proportional to the splitting (Melnikov quantity) times a constant depending on the number of tori.

The proof of theorem 4 follows from lemma 17 by induction. □

VIII ABOUT THE ASYMPTOTIC INSTABILITY CONJECTURE

All our approach extend to near integrable Hamiltonian systems. Let $\Lambda = k_0\mathbb{Z}$ be a sub lattice of $\mathbb{Z}^n$ generated by $k_0 \in \mathbb{Z}^n_K$, where $K > 0$ and

$$\mathbb{Z}^n_K = \{k \in \mathbb{Z}^{n+1}, \mid k \mid \leq K\}. \tag{53}$$

Let $\alpha > 0$, a subset $D \subset \mathbb{R}^n$ is said to be α, K-nonresonant modulo Λ, if at every point $I \in D$, $\mid k.\omega(I) \mid \leq \alpha$ for all $k \in \mathbb{Z}^n_K \setminus \Lambda$.

In the vincinity of D the perturbed Hamiltonian (1) can be put into a Λ-resonant normal form $h(I)+g(I,\theta)+f_*(I,\theta)$ where

$$g(I,\theta) = \sum_{k\in\Lambda} g_k(I)e^{ik.\theta}, \tag{54}$$

contains only Λ-resonant terms, while f_* is a general term.

By mean of a unimodular transformation in the phase space, we can always assume that Λ is generated by the first vector of the basis in the frequency space.

There exists an analytic canonical coordinate systems ([36], [6], [16], [33]) such that the Hamiltonian takes the form

$$H_{\epsilon,\mu}(I,\theta) = h(I) + \epsilon g(I,\theta_1) + \epsilon\mu f_*(I,\theta), \tag{55}$$

where $\mu = e^{-Ks/6}$ and s depends on the domains of analyticity of h in the angle variables.

Under generic non degeneracy conditions ([36], assumption 3, p.203 and comments below), for fixed I and ϵ, g is a one degree of freedom Hamiltonian system whose phase portrait is qualitatively that of a simple pendulum. In particular, it has hyperbolic equilibria connected by homoclinic orbits.

We can then defined a hyperbolic set associated to a simple resonance as in section IV, called *simple resonance manifold*. The study of the geometry and dynamics of this set follows as in section IV. Then, all our approach, based on transversal homoclinic hyperbolic tori and connectors generalizes when $n = 3$.

As μ depends on K, we will have to construct, for a given ϵ, a set $\mathcal{R}_1(\epsilon)$ (see section II C) which corresponds to simple resonances with $K > c(\epsilon)$ where c is a suitable function of ϵ (like $c(\epsilon) = \frac{1}{\epsilon^{2n}}$.

This approach is connected with the classical proof of the *Nekhoroshev* theorem ([32], [35]), which asserts that $\mid I(t) - I(0) \mid < \epsilon^a$ for $\mid t \mid < \left(\frac{1}{\epsilon}\right) e^{\frac{1}{\epsilon^b}}$. If h is convex then one can take $a = b = 1/2n$ ([35], [23]).

For $n \geq 4$, connectors must be constructed via a controled Arnold diffusion on the simple resonance manifold.

Then, via a simple induction on the number of degrees of freedom, one can proved that there exists orbits of instability along the set $\mathcal{R}_1(\epsilon)$, for all $0 < \epsilon < \epsilon_0$ sufficiently small.

In order to prove completely the asymptotic instability conjecture, we needs to understand the dynamics in the vincinity of double resonances [2]. In particular, we must be able to construct orbits which drift from a given simple resonance to another. We refer to ([10], appendix A) for a first example and to [27] for a first step toward this problem.

Via this last remark, our approach coupled with the dynamics in the neighborhood of a double resonance provides a general scheme to prove the asymptotic instability conjecture. A detailed description of this scheme will appear in a subsequent paper [11].

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