A NOTION OF STOCHASTIC CHARACTERISTICS FOR PARABOLIC AND MIXED-TYPE PARTIAL DIFFERENTIAL EQUATIONS

by

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Abstract. — The classical method of characteristics does not apply to parabolic or mixed type PDEs. Using the formalism of stochastic embedding of dynamical systems initiated in [Cresson-Darses, J. Math. Phys. 48, 072703 (2007)], we define a notion of stochastic characteristics. We then look for stochastic characteristics of some classical parabolic and mixed type PDEs : the Burger's equation and the Heat equation. In particular, we prove that the stochastic characteristics of the Burger's equation and the Heat equation correspond to (weak) critical points of an explicit stochastic Lagrangian functional over particular sets of stochastic processes.

Contents

1. Introduction

The classical method of characteristics for a PDE is to look for curves $s \mapsto (x(s), t(s))$ where $x(s)$ and $t(s)$ are solutions of an ordinary differential equation such that solutions $u(x,t)$ of the PDE satisfy

$$\frac{d}{ds}\left(u(x(s),t(s))\right) = F(x(s),t(s)),$$

where F is the non homogeneous part of the PDE.

In many cases, we can choose

$$\frac{dt}{ds} = 1$$

so that one is reduced to find a curve $t \to x(t)$ satisfying the following ordinary differential equation

$$\frac{d}{dt}\left(u(x(t),t)\right) = F(x(t),t).$$

The method of characteristics does not work for parabolic PDE's and PDE's of mixed type like elliptic/parabolic (as for example the transport equation with diffusion).

Using the formalism of stochastic embedding of dynamical systems introduced in [**1**], we develop a notion of (time reversal) stochastic characteristics extending the classical notion. The definitions are given in Section 2.2. In Section 3, we give examples of parabolic and mixed type PDEs, the Burger's equation and the Heat equation respectively, for which we can characterize the (time reversal) stochastic characteristics. In Section 4, we prove that the (time reversal) stochastic characteristics of the Burger's equation and the Heat equation correspond to (weak) critical point of an explicit stochastic Lagrangian functional. The class of stochastic processes under consideration for each equation is different and explicitly defined. In Section 5, we discuss some perspectives of our approach.

Related works concerning the use of the stochastic calculus of variations in the sense of [**1**] in order to derive results about classical PDEs can be found in the articles of Yasue and co-workers ([**17**],[**13**]).

2. Toward a notion of stochastic characteristics for PDEs

2.1. Reminder on stochastic derivatives. —

2.1.1. Notations. — Let $T > 0$, $\nu > 0$ and $d \in \mathbb{N}^*$. Let $\mathbb{L}$ be the set of all measurable functions $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the following hypothesis: There exists $K > 0$ such that for all $x, y \in \mathbb{R}^d$: $\sup_t |f(t,x) - f(t,y)| \le K\,|x-y|$ et $\sup_t |f(t,x)| \le K(1+|x|)$.

We are given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which a family $(W^{(b,\sigma)})_{(b,\sigma)\in\mathbb{L}\times\mathbb{L}}$ of Brownian motions indexed by $\mathbb{L}\times\mathbb{L}$ is defined. If $b,\sigma\in\mathbb{L}$, we denote by $\mathcal{P}^{(b,\sigma)}$ the natural filtration associated to $W^{(b,\sigma)}$. Let $\mathcal{P}$ be the filtration generated by the filtrations $\mathcal{P}^{(b,\sigma)}$ where $(b,\sigma)\in\mathbb{L}\times\mathbb{L}$, and we set: For $t\in[0,T]$,

$$\mathcal{P}_t = \bigvee_{(b,\sigma)\in\mathbb{L}\times\mathbb{L}} \mathcal{P}_t^{(b,\sigma)}.$$

Let $F([0,T]\times\mathbb{R}^d)$ be the space of measurable functions defined on $[0,T]\times\mathbb{R}^d$ and let $F([0,T]\times\Omega)$ be the space of measurable stochastic processes defined on $[0,T]\times\Omega$.

Let us define the involution $\phi : F([0,T]\times\mathbb{R}^d)\to F([0,T]\times\mathbb{R}^d)$ such that for all $t\in[0,T]$ and $x\in\mathbb{R}^d$, $(\phi u)(t,x) = -u(T-t,x)$. We also define the time-reversal involution on stochastic processes: $r : F([0,T]\times\Omega)\to F([0,T]\times\Omega)$, $r(X)_t(\omega) = X_{T-t}(\omega)$.

It is convenient to use the bar symbol to denote these two involutions. We now agree to denote deterministic functions by small letters and stochastic processes by capital letters. So there will not be any confusion when using the bar symbol: $\overline{u} := \phi u$ and $\overline{X} := r(X)$.

The space $\mathbb{R}^d$ is endowed with its canonical scalar product $\langle\cdot,\cdot\rangle$. Let $|\cdot|$ be the induced norm. If $f : [0,T]\times\mathbb{R}^d\to\mathbb{R}$ is a smooth function, we set $\partial_j f = \frac{\partial f}{\partial x_j}$. We denote by $\nabla f = (\partial_i f)_i$ the gradient of f and by $\Delta f = \sum_j \partial_j^2 f$ its Laplacian. For a smooth map $\Phi : [0,T]\times\mathbb{R}^d\to\mathbb{R}^d$, we denote by Φ^j its j^{th}-component, by $(\partial_x\Phi)$ its differential which we represent into the canonical basis of $\mathbb{R}^d$: $(\partial_x\Phi) = (\partial_j\Phi^i)_{i,j}$, and by $\nabla\cdot\Phi = \sum_j\partial_j\Phi^j$ its divergence. By convention, we denote by $\Delta\Phi$ the vector $(\Delta\Phi^j)_j$. The notation $(\Phi\cdot\nabla)\Phi$ denotes the parallel derivative of Φ along itself, whose coordinates are: $((\Phi\cdot\nabla)\Phi)^i = \sum_j \Phi^j\partial_j\Phi^i$. The image of a vector $u\in\mathbb{R}^d$ under a linear map M is simply denoted by Mu, for instance $(\partial_x\Phi)u$. A map $a : [0,T]\times\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d$ is viewed as a $d\times d$ matrix whose columns are denoted by a_k. Finally, we denote by $\nabla\cdot a$ the vector $(\nabla\cdot a_k)_k$.

2.1.2. Stochastic derivatives. —

Definition 1. — *We denote by Λ^1 the space of all diffusions X satisfying the following conditions:*

(i) *X is a solution on $[0,T]$ of the SDE: $dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t^{(b,\sigma)}$, $\quad X_0 = X^0$ where $X^0\in L^2(\Omega)$ and $(b,\sigma)\in\mathbb{L}\times\mathbb{L}$,*

(ii) *For all $t\in(0,T)$, X_t admits a density $p_t(\cdot)$,*

(iii) *Setting $a^{ij} = (\sigma\sigma^*)^{ij}$, for all $i\in\{1,\cdots,n\}$, $t_0>0$,*

$$\int_{t_0}^T\int_{\mathbb{R}^d}\left|\partial_j(a^{ij}(t,x)p_t(x))\right|dxdt < +\infty,$$

(iv) *For all i,j,t,*

$$\frac{\partial_j(a^{ij}(t,\cdot)p_t(\cdot))}{p_t(\cdot)} \in \text{Lipschitz}.$$

A diffusion verifying (i) will be called a (b,σ)-diffusion and we denote it by $X^{(b,\sigma)}$.

We denote by Λ^1_v the closure of $\mathrm{Vect}(\Lambda^1)$ in $L^1(\Omega\times[0,T])$ endowed with the usual norm $\|\cdot\| = E\int|\cdot|$.

We know that the reversed process of any element Λ^1 is still a Brownian diffusion driven by a Brownian motion $\widehat{W}^{(b,\sigma)}$ (*cf* [**14**], Theorem 2.3). We denote by $\widehat{\mathcal{P}}^{(b,\sigma)}$ the natural filtration associated to $\widehat{W}^{(b,\sigma)}$. Let $\widehat{\mathcal{P}}$ be the filtration defined by: For all $t\in[0,T]$,

$$\widehat{\mathcal{P}}_t = \bigvee_{(b,\sigma)\in\mathbb{L}\times\mathbb{L}} \widehat{\mathcal{P}}^{(b,\sigma)}_t.$$

We finally consider the filtration $\mathcal{F}$ such that $\mathcal{F}_t = \widehat{\mathcal{P}}_{T-t}$ for all $t\in[0,T]$.

Proposition 1 **(Definition of Nelson Stochastic derivatives)**
Let $X = X^{b,\sigma}\in\Lambda^1$ with $a^{ij} = (\sigma\sigma^*)^{ij}$ and $a^j = (a^{1j},\cdots,a^{dj})$. For almost all $t\in(0,T)$, the Nelson stochastic derivatives exist in $L^2(\Omega)$:

$$(1)\qquad DX_t := \lim_{h\to 0^+} E\left[\frac{X_{t+h}-X_t}{h} \mid \mathcal{P}_t\right] = b(t,X_t)$$

$$(2)\qquad D_*X_t := \lim_{h\to 0^+} E\left[\frac{X_t-X_{t-h}}{h} \mid \mathcal{F}_t\right] = b(t,X_t) - \frac{1}{p_t(X_t)}\sum_j \partial_j(a^j(t,X_t)p_t(X_t)).$$

Therefore, for $X^{b,\sigma}\in\Lambda^1$, there exists a measurable function b_* such that $D_*X_t = b_*(t,X_t)$. We call it the left velocity field of X. It turns out to be an important objet for the sequel. Also, this field is related to the drift of the time reversed process $\overline{X}$ through the following identity:

$$(3)\qquad D\overline{X} = \overline{b_*}.$$

We denote by $\Lambda^2 = \{X\in\Lambda^1; DX,\ D_*X\in\Lambda^1\}$ and Λ^2_v the closure of $\mathrm{Vect}(\Lambda^2)$ in $L^1(\Omega\times[0,T])$. We then define Λ^k_v in an obvious way.

We denote by $\mathcal{D}_\mu$ the stochastic derivative introduced in [**1**] and defined by

$$(4)\qquad \mathcal{D}_\mu = \frac{D+D_*}{2} + \mu\frac{D-D_*}{2},\quad \mu\in\{0,\pm1,\pm i\}.$$

We can extend $\mathcal{D}$ by $\mathbb{C}$-linearity to complex processes $\Lambda^1_{\mathbb{C}} := \Lambda^1_v \oplus i\Lambda^1_v$.

Theorem 1. — *Let $X^{(b,\sigma)}\in\Lambda^1$, $a=\sigma\sigma^*$ and $f\in C^{1,2}(I\times\mathbb{R}^d)$ such that $\partial_t f$, ∇f and $\partial_{ij}f$ are bounded. We get:*

$$(5)\qquad \mathcal{D}_\mu f(t,X^{(b,\sigma)}_t) = \left(\partial_t f + \mathcal{D}X^{(b,\sigma)}_t\cdot\nabla f + \frac{\mu}{2}\sum_{k,j} a^{kj}\partial_{kj}f\right)(t,X^{(b,\sigma)}_t).$$

We refer to [**15**] or ([**5**], Proposition 3,p.394) for a proof.

2.2. A notion of stochastic characteristics. — Following the general strategy of the stochastic embedding formalism developed in [**1**], a natural idea is to replace the classical curve $s \to x(s)$ by a stochastic process $s \to X_s$ and the classical derivative by the stochastic derivative $\mathcal{D}_\mu$. We are then leaded to the following notion of stochastic characteristics :

Definition 2 **(Stochastic characteristics)**. — *We say that a stochastic process $s \to X_s$ is a stochastic characteristic for a given PDE if the solution $u(x,t)$ satisfies*

$$\mathcal{D}_\mu \left(u(X_s, s)\right) = F(X_s, s),$$

and X_s satisfies an ordinary differential equation in $\mathcal{D}_\mu$.

Stochastic characteristics do not always exist. We will see on some examples that there exist obstructions at least if we restrict our attention to specific class of stochastic processes like diffusion processes. However, we introduce a second notion of stochastic characteristics called time-reversal stochastic characteristics which can be found even when stochastic characteristics do not exist.

Definition 3 **(Time reversal stochastic characteristics)**. — *We say that a stochastic process $s \to X_s$ is a time reversal stochastic characteristic for a given PDE if the solution $u(x,t)$ satisfies*

$$\mathcal{D}_\mu \left(\bar{u}(X_s, s)\right) = F(X_s, s),$$

and X_s satisfies an ordinary differential equation in $\mathcal{D}_\mu$.

In the following Sections, we look for classical and time reversal stochastic characteristics of the Burger's equation and the Heat equation. In particular, we prove that they correspond to critical points of an explicit stochastic Lagrangian system.

Remark 1. — *The previous notion of stochastic characteristics for classical PDEs shares no relations with the notion of stochastic introduced for example by N. V. Krylov and B.L. Rozovskii in* [**11**] *for stochastic PDEs.*

2.3. Searching for stochastic characteristic of PDEs. — By the Definition 2.2, a ($\mathcal{D}_\mu$) stochastic characteristic of an homogeneous partial differential equation must satisfy

$$\mathcal{D}_\mu u(X_t, t) = 0, \tag{6}$$

for all solutions $u(x,t)$ of the PDE.

Assuming that the solution u belongs to $C^{1,2}(I\times\mathbb{R}^d)$ and using Theorem 5, equation (6) is equivalent to

$$\mathcal{D}_\mu u(t,X_t^{(b,\sigma)})=\left(\partial_t u+\mathcal{D}_\mu X_t^{(b,\sigma)}\cdot\nabla u+\frac{\mu}{2}\sum_{k,j}a^{kj}\partial_{kj}u\right)(t,X_t^{(b,\sigma)}). \tag{7}$$

In order to simplify this expression we restrict our attention to stochastic characteristics belonging to the set of diffusion processes with constant diffusion. Precisely, we introduce the following space :

Definition 4. — *We denote by Λ_c^2 the set of stochastic processes composed of all the diffusions $X^{(b,\sigma)}$ such that $\sigma=c$ is constant and the drift b is C^2, bounded with all its second derivatives bounded, and $\nabla\log\rho_t$ has bounded second order derivatives.*

We can prove that $\Lambda_c^2\subset\Lambda^2$ thanks to Prop. 2 p.394 in [**5**]. Recall that a (u,σ)-diffusion with a constant diffusion coefficient σ is of the form:

$$X_t=X_0+\int_0^t b(s,X_s)ds+\sigma W_t. \tag{8}$$

As a consequence, the previous formula for stochastic characteristics over Λ_c^2 is given by

$$\mathcal{D}_\mu u(t,X_t^{(b,\sigma)})=\left(\partial_t u+\mathcal{D}_\mu X_t^{(b,\sigma)}\cdot\nabla u+\frac{\mu}{2}\sigma^2\nabla u\right)(t,X_t^{(b,\sigma)}). \tag{9}$$

The main problem is now to identify the subset of Λ_c^2 which corresponds to stochastic characteristic for a given homogeneous PDE.

In the time reversal case, we obtain the same formula replacing u by $\bar{u}$:

$$\mathcal{D}_\mu \bar{u}(t,X_t^{(b,\sigma)})=\left(\partial_t \bar{u}+\mathcal{D}_\mu X_t^{(b,\sigma)}\cdot\nabla \bar{u}+\frac{\mu}{2}\sigma^2\nabla \bar{u}\right)(t,X_t^{(b,\sigma)}). \tag{10}$$

The following sections are devoted to finding (time reversal) stochastic characteristic for the Burger's equation and the Heat equation.

3. Application to parabolic and mixed-type PDEs

In this section, we search for (time reversal) stochastic characteristics of parabolic and mixed-type PDEs. As an example of mixed-type and parabolic PDEs, we consider the Burger's equation and the Heat equation respectively for which we give an explicit characterization of the (time reversal) stochastic characteristics.

3.1. The Burger's equation. — The Burger's equation can be written as

$$\partial_t u + (u \cdot \nabla) u = \nu \Delta u, \tag{11}$$

where $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the velocity field and $\nu > 0$ is the viscosity.

3.1.1. Stochastic characteristics. — Using the formula (9), it is easy to find a class of stochastic processes in Λ_c^2 which are stochastic characteristic of the Burgers's equation. Indeed, let us assumes that a (b,σ) diffusion of Λ_c^2 satisfies the following conditions

$$\begin{cases} \mathcal{D}_\mu X_t^{(b,\sigma)} & = & u(t, X_t), \\ \frac{\mu}{2}\sigma^2 & = & -\nu, \end{cases} \tag{12}$$

then it is a stochastic characteristic of the Burger's equation.

The existence of such processes depends on the value of μ as already seen by the second condition of (12). Indeed, we must have

$$\mu < 0, \tag{13}$$

in order to satisfy (12). We then obtain the following lemma :

Lemma 1. — *A $\mathcal{D}_\mu$ stochastic characteristic of the Burger's equation exists if and only if $\mu = -1$, i.e. $\mathcal{D}_{-1} = D_*$.*

We assume now that $\mu = -1$. The second condition of (12) fixes the value of the constant diffusion coefficient σ to be

$$\sigma = \pm\sqrt{2\nu}. \tag{14}$$

Remark 2. — *It must be pointed out that the previous obstruction is not valid when $\nu = 0$. In this case of course, we can consider $\mu = 1$ and deterministic processes $X_t \in \Lambda_c^2$ of the form*

$$X_t = X_0 + \int_0^t u(s, X_s)\, ds. \tag{15}$$

We then recover the classical definition of characteristic for a PDE. The Burger's equation reduces to the inviscid Burger's equation which is a simple example of a non linear hyperbolic PDE. For this equation, the classical method of characteristic can be applied.

Remark 3. — *Another possibility which we have not explored is to take $\mu = 1$ and to consider a complex diffusion coefficient. The notion of complex diffusion has been used in the literature for rigorous foundations of the Feynmann Paths Integrals and several approaches to Lagrangian variational formulation of the Schrödinger equation. Nevertheless, we have not find a satisfying place where such a notion is mathematically well defined. The second problem is to interpret the complex diffusion coefficient.*

We can solve the first equation of (12) using the formula (2) for a diffusion process $X_t \in \Lambda_c^2$ and $\mu = -1$. We have

$$D_* X_t = b(t, X_t) - \sigma^2 \frac{\nabla p_t}{p_t}(X_t). \tag{16}$$

As $D_* X_t = u(t, X_t)$, we obtain

$$b(t, X_t) = u(t, X_t) + \sigma^2 \frac{\nabla p_t}{p_t}(X_t). \tag{17}$$

We then have the following theorem :

Theorem 2 **(Stochastic characteristics of the Burger's equation)**
The stochastic characteristics $X_t \in \Lambda_c^2$ of the Burger's equation are of the form

$$X_t = X_0 + \int_0^t \left(u(s, X_s) + \sigma^2 \frac{\nabla p_s}{p_s}(X_s) \right) ds + \sqrt{2\nu} dW_t, \tag{18}$$

where u is a solution of the Burger's equation.

This result is not satisfying because the definition of the drift does not only depend on u but also on the probability density p_t of the process. The density is also linked with the drift via the Fokker-Planck equation leading to a difficult system of partial differential equations. In the next Section, we then consider time reversal stochastic characteristics which are more simply characterized.

3.1.2. Time reversal stochastic characteristics. — We first rewrite the Burger's equation in term of $\bar{u}$. A simple computation leads to

$$\partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nu \Delta \bar{u} = 0. \tag{19}$$

Let $X_t \in \Lambda_c^2$ be a (b, σ) diffusion satisfying the following conditions

$$\left\{ \begin{array}{lcl} \mathcal{D}_\mu X_t^{(b,\sigma)} & = & \bar{u}(t, X_t), \\ \frac{\mu}{2}\sigma^2 & = & \nu. \end{array} \right. \tag{20}$$

Then it is a time-reversal stochastic characteristic of the Burger's equation.

The existence of such processes depends on the value of μ as already seen by the second condition of (20). Indeed, we must have

$$\mu > 0, \tag{21}$$

in order to satisfy (20). We then obtain the following lemma :

Lemma 2. — *A $\mathcal{D}_\mu$ time reversal stochastic characteristic of the Burger's equation exists if and only if $\mu = 1$, i.e. $\mathcal{D}_1 = D$.*

We assume now that $\mu = 1$. The second condition of (20) fixes the value of the constant diffusion coefficient σ to be

$$\sigma = \pm\sqrt{2\nu}. \tag{22}$$

The first condition of (20) fixes the drift part of the time reversal stochastic characteristic to be $\bar{u}$. We then obtain the following Theorem.

Theorem 3. — *[Time reversal stochastic characteristics for the Burger's equation] Time reversal stochastic characteristic $X_t \in \Lambda_c^2$ of the Burger's equation are given by*

$$X_t = X_0 + \int_0^t \bar{u}(s, X_s)ds + \sqrt{2\nu}W_t, \tag{23}$$

where u is a solution of the Burger's equation.

The viscosity coefficient plays an important role in this result as it controls the diffusion coefficient of the stochastic process. In particular, for the inviscid Burger's equation corresponding to $\nu = 0$, the time reversal stochastic processes reduce to deterministic processes and then to classical characteristic.

Of course, one is interested in a more intrinsic characterization of the time-reversal stochastic characteristics. We prove in the next section that they correspond to critical point of an explicit stochastic Lagrangian functional in the sense of [**1**].

3.2. The Heat equation. — The Heat equation is a very simple parabolic PDE given by

$$\partial_t u - \nu\Delta u = 0, \tag{24}$$

where the viscosity coefficient $\nu > 0$.

As in the previous Section, we can search for (time reversal) stochastic characteristics of the Heat equation.

We consider stochastic processes of the form:

$$X_t = X_0 + \sigma B_t, \tag{25}$$

where B_t is a Brownian bridge pinned to be 0 at T.

Applying the stochastic derivative $\mathcal{D}_\mu$, we obtain

$$\mathcal{D}_\mu u(t, \sigma B_t) = \left(\partial_t u + \sigma \mathcal{D}_\mu B_t.\nabla u + \mu \frac{\sigma^2}{2}\Delta u\right)(t, \sigma B_t). \tag{26}$$

In order to have $-\nu$ in front of the Laplacian of u, we take

$$\mu = -1, \quad \sigma = \sqrt{2\nu}. \tag{27}$$

A good property of the Brownian Bridge is that it satisfies

$$D_* B_t = 0. \tag{28}$$

As a consequence, we have the following solution for the stochastic characteristics of the Heat equation :

Theorem 4. — *The family of stochastic processes (25) provides stochastic characteristics for the Heat equation.*

The situation is then completely different than for the Burgers equation. The underlying set of stochastic characteristics has no connections with the time reversal stochastic characteristics of the Burgers equation. The fact to remove the convection term in the Burgers equation has strong consequences on the structure of the set of solutions. This is a well known phenomenon in PDEs. We refer in particular to [**19**] for an interesting study of this phenomenon for the Navier-Stokes equation.

Remark 4. — *The Burger's equation and the Heat equation are related by the well known Hopf-Cole transformation (see* [**12**],[**2**]*). It can be interesting to look for the behaviour of the set of stochastic characteristics with respect to this transformation.*

4. Stochastic characteristics and stochastic Lagrangian systems

In this Section, we give an alternative characterization of (time reversal) stochastic characteristics obtain for the Burger's and the Heat equation in term of stochastic Lagrangian systems.

4.1. Reminder about stochastic Lagrangian systems.

— We follow our previous work [**1**] to which we refer for complete proofs (see also the work of K. Yasue [**16**]).

A stochastic Lagrangian functional is defined as follow.

Definition 5. — *Let L be an admissible Lagrangian function. Set*

$$\Xi = \left\{ X \in \Lambda^1, E\left[\int_0^T |L(X_t, \mathcal{D}_\mu X_t)| dt \right] < \infty \right\}.$$

The functional associated to L is defined by

$$F : \left\{ \begin{array}{ccc} \Xi & \longrightarrow & \mathbb{C} \\ X & \longmapsto & E\left[\int_0^T L(X_t, \mathcal{D}_\mu X_t) dt \right] \end{array} \right. . \tag{29}$$

In what follows, we need a special notion which we will call L-adaptation:

Definition 6. — *Let L be an admissible Lagrangian function. A process $X \in \Lambda^1$ is said to be L-adapted if:*

(i) $X \in \Xi$;
(ii) *For all $t \in I$, $\partial_x L(X_t, \mathcal{D}_\mu X_t) \in L^2(\Omega)$;*
(iii) $\partial_v L(X_t, \mathcal{D}_\mu X_t) \in \Lambda^1$.

The set of all $L-$adapted processes will be denoted by $\mathcal{L}$.

We introduce the following terminology:

Definition 7. — *Let Γ be a subspace of Λ^1 and $X \in \Lambda^1$. A Γ-variation of X is a stochastic process of the form $X + Z$, where $Z \in \Gamma$. Moreover set*

$$\Gamma_\Xi = \{Z \in \Gamma, \forall X \in \Xi, Z + X \in \Xi\}.$$

We now define a notion of *differentiable functional.* Let Γ be a subspace of Λ^1.

Definition 8. — *Let L be an admissible Lagrangian function and F the associated functional. The functional F is called Γ-differentiable at $X \in \mathcal{L}$ if for all $Z \in \Gamma_\Xi$*

$$F(X + Z) - F(X) = dF(X, Z) + R(X, Z), \tag{30}$$

where $dF(X, Z)$ is a linear functional of $Z \in \Gamma_\Xi$ and $R(X, Z) = o(\| Z \|)$.

A Γ-critical process for the functional F is a stochastic process $X \in \Xi \cap \mathcal{L}$ such that $dF(X, Z) = 0$ for all $Z \in \Gamma_\Xi$ such that $Z(a) = Z(b) = 0$.

The main result of [**1**] is the following analogue of the least-action principle for Lagrangian mechanics.

***Theorem 5* (Global Least action principle)**. — *Let L be an admissible lagrangian with all second derivatives bounded. A necessary and sufficient condition for a process $X \in \mathcal{L} \cap \Lambda^3$ to be a Λ^1-critical process of the associated functional F is that it satisfies*

$$\frac{\partial L}{\partial x}(X_t, \mathcal{D}_\mu X_t) - \mathcal{D}_{-\mu}\left[\frac{\partial L}{\partial v}(X_t, \mathcal{D}_\mu X_t)\right] = 0. \tag{31}$$

We call this equation the Global Stochastic Euler-Lagrange equation (GSEL).

We refer to ([**1**], Theorem 3.1, p.33-34) for a proof.

The main drawback of Theorem 5 is that equation (31) is not *coherent* (see [**1**],§.6.2) *i.e.* that it does not coincide with the direct stochastic embedding of the classical Euler-Lagrange equation of the form

$$\frac{\partial L}{\partial x}(X_t, \mathcal{D}_\mu X_t) - \mathcal{D}_\mu \left[\frac{\partial L}{\partial v}(X_t, \mathcal{D}_\mu X_t) \right] = 0. \tag{32}$$

except when $\mu = 0$.

In order to obtain a coherent embedding without imposing $\mu = 0$, we must restrict the set of variations. Let us introduce the space of Nelson differentiable processes:

$$\mathcal{N}^1 = \{X \in \Lambda^1, DX = D_* X\}. \tag{33}$$

Remark 5. — *The most simple examples of Nelson's differentiable processes are given by differentiable deterministic processes (see* [**1**]*,§.3.2.2.1). A more involved one is provided by the solution of the stochastic harmonic oscillator defined by the system*

$$\left\{ \begin{array}{lcl} dX(t) & = & V(t)dt, \\ dV(t) & = & -\alpha V(t)dt - \omega^2 X(t)dt + \sigma dW_t, \\ X(0) = X_0, & & V(0) = V_0. \end{array} \right. \tag{34}$$

In this case, we have (see [**1**]*,§.3.2.2.2) that* $DX_t = D_* X_t = V(t)$*.*

More involved examples are provided by solutions of stochastic differential equations driven by a fractional Brownian motion with a Hurst index $H > 1/2$ *(see* [**6**]*, Theorem 20).*

Using $\mathcal{N}^1$-variations we have been able to prove the following result [**1**]:

Proposition 2. — *Let* L *be an admissible lagrangian with all second derivatives bounded. A solution of the equation*

$$\frac{\partial L}{\partial x}(X_t, \mathcal{D}_\mu X_t) - \mathcal{D}_\mu \left[\frac{\partial L}{\partial v}(X_t, \mathcal{D}_\mu X_t) \right] = 0, \tag{35}$$

called the Stochastic Euler-Lagrange Equation (SEL), is a $\mathcal{N}^1$*-critical process for the functional* F *associated to* L*.*

We refer to ([**1**], Lemma 3.4, p.34) for a proof.

We have not been able to prove the converse of this lemma for $\mathcal{N}^1$-variations.

4.2. Lagrangian stochastic characteristics. — In this Section, we consider the Lagrangian

$$L(t, x, v) = \frac{v^2}{2}. \tag{36}$$

The associated stochastic Lagrangian functional is given by

$$\mathcal{L}(X_t) = E\left[\int_a^b \frac{1}{2}(\mathcal{D}_\mu X_s)^2\, ds\right]. \tag{37}$$

4.2.1. Lagrangian stochastic characteristics. — Applying Theorem 5, a (b,σ)-diffusion X_t is a critical point of the natural Lagrangian (36) for full variations, if and only if it is a critical point of the Lagrangian (36), i.e.

$$\mathcal{D}_{-\mu}\left(\mathcal{D}_\mu X_t\right) = 0. \tag{38}$$

Definition 9. — *A (time reversal) stochastic characteristic X_t satisfying equation (38) is called a Lagrangian (time reversal) stochastic characteristic.*

As a consequence, a $\mathcal{D}_{-\mu}$ stochastic characteristic X_t (resp. time reversal stochastic characteristic) of a given homogeneous PDE is also a critical point of the stochastic Lagrangian functional (37) if it satisfies

$$\mathcal{D}_\mu X_t = \pm u(t,X_t), \quad (\text{resp. } \mathcal{D}_\mu X_t = \pm \bar{u}(t,X_t)\,), \tag{39}$$

where u is a solution of the PDE.

A stochastic characteristic (resp. time reversal stochastic characteristic) must satisfy some particular constraints like conditions (9) (resp. conditions (20)) for the Burger's equation in order to correspond to a critical point of a stochastic Lagrangian functional. We will see in the following Section that the resulting system of equations does not always possess a solution.

4.2.2. Weak Lagrangian stochastic chracteristics. — Applying Proposition 2, a (b,σ)-diffusion X_t solution of the equation

$$\mathcal{D}_\mu\left(\mathcal{D}_\mu X_t\right) = 0. \tag{40}$$

is a weak critical point of the natural Lagrangian (36).

Definition 10. — *A (time reversal) stochastic characteristic X_t satisfying equation (40) is called a weak Lagrangian (time reversal) stochastic characteristic.*

We will see in the next Section that some stochastic characteristics of PDEs can be weak Lagrangian but not a Lagrangian one.

A $\mathcal{D}_\mu$ weak stochastic characteristic X_t (resp. weak time reversal stochastic characteristic) of a given homogeneous PDE is also a critical point of the stochastic Lagrangian functional (37) if it satisfies condition (39).

4.3. Lagrangian stochastic characteristics of the Burger's equation. —

4.3.1. Stochastic characteristics. — Using conditions (12), the Lagrangian stochastic characteristics of the Burger's equation must satisfy the following system

$$\begin{array}{rcl} D_* X_t & = & u(t, X_t), \\ D X_t & = & \pm u(t, X_t). \end{array} \tag{41}$$

We have two cases :

- If $DX_t = u(t, X_t)$ then $DX_t = D_* X_t$ and X_t is Nelson differentiable process in the sense of [**1**]. By ([**1**], Lemma 1.9, p.20) and the fact that $X_t \in \Lambda_c^2$ i.e. has a constant diffusion coefficient, we conclude that X_t can not exist.
- If $DX_t = -u(t, X_t)$ then $DX_t = -D_* X_t$. Following the work of S. Darses and I. Nourdin [**5**] this impose strong constraints on the drift coefficient (see [**5**], Proposition 4, p. 396 in the case of a homogeneous drift).

4.3.2. Time reversal stochastic characteristics. — Using conditions (20), the Lagrangian time reversal stochastic characteristics of the Burger's equation must satisfy the following system

$$\begin{array}{rcl} D X_t & = & \bar{u}(t, X_t), \\ D_* X_t & = & \pm \bar{u}(t, X_t). \end{array} \tag{42}$$

We have the same conclusions as in the previous Section.

4.4. Weak Lagrangian stochastic characteristics of the Burger's equation. —

4.4.1. Weak Lagrangian stochastic characteristics. — Following Lemma 1 Weak Lagrangian stochastic characteristics of the Burger's equation satisfy

$$D_* X_t = \pm u(t, X_t), \tag{43}$$

which is indeed the case as $D_* X_t = u(t, X_t)$ be Theorem 2.

Theorem 6. — *Weak stochastic characteristics of the Burger's equation given by Theorem 2 are weak Lagrangian time reversal stochastic characteristics.*

4.4.2. Weak Lagrangian time reversal stochastic characteristics. — Following Lemma 2 a weak Lagrangian time reversal stochastic characteristic must satisfy the following system

$$D X_t = \pm \bar{u}(t, X_t), \tag{44}$$

which is indeed the case as $DX_t = u(t, X_t)$ by Theorem 3. We then have :

Theorem 7. — *Weak time reversal stochastic characteristics of the Burger's equation given by Theorem 3 are weak Lagrangian time reversal stochastic characteristics.*

4.5. Lagrangian stochastic characteristics of the Heat equation. — As for the existence of stochastic characteristics, we obtain a stronger result concerning the Lagrangian structure of the solutions.

Theorem 8. — *Let X_t be a process of the form:*

$$dX_t = u(t, \sqrt{2\nu} B_t) dt. \tag{45}$$

The velocity field u satisfies the Heat equation if and only if the stochastic process X_t is a Λ^1 critical process of the stochastic functional

$$X \mapsto \mathrm{E}\left[\int_0^T L(\sqrt{2\nu} B_t, DX_t) dt\right] \tag{46}$$

where $L(x,v) = \dfrac{v^2}{2}$.

Proof. — The Λ^1 least action principle reads:

$$D_* DX_t = 0.$$

But

$$D_* DX_t = \Big(\partial_t u + \sqrt{2\nu}\ (\partial_x u) D_* B_t - \nu \Delta u\Big)(t, \sqrt{2\nu} B_t). \tag{47}$$

Since $D_* B = 0$ and the probability density of the bridge is everywhere positive, we can deduce the desired equivalence. □

Remark 6. — *This result is stronger than the previous result on weak Lagrangian (time reversal) stochastic characteristics of the Burgers equation as we have a complete equivalence through the global least action principle.*

5. Conclusion and perspectives

The notion of stochastic characteristics introduced in this paper cover parabolic and mixed type PDEs. The previous approach and results can be developed in the following directions :

- Although we have obtain a characterization of the stochastic characteristics for the Burger's equation as (weak) critical points of an explicit stochastic Lagrangian functional, we have not been able to prove the existence of such processes directly from the stochastic equation. The results and tools developed by A. Cruzeiro and E. Sharmarova in ([**3**],[**4**]) can provide a way to solve this problem.
- The classical characteristics method allows to reconstruct the solutions of the PDE knowing the characteristics. The same strategy must be develop in our setting in order to see if one can reconstruct the (classical) solutions of the PDE from the stochastic characteristics.

– The restriction on the class of PDEs that we can study in this paper is largely due to the use of the Nelson's stochastic derivatives. A more general theory can probably be developed using for example the notion of stochastic derivatives for fractional diffusions defined by S. Darses and I. Nourdin in [**6**] for which many computations can again be made (see [**7**]). As a more abstract level, one can also use the notion of differentiating sigma-fields introduced by S. Darses, I. Nourdin and G. Peccati in [**8**].

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