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Fractional differential equations and the Schrödinger equation

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Abstract

In a previous paper, we defined, following a previous work of Kolvankar and Gangal, a notion of α-derivative, $0 < \alpha < 1$. In this article, we study α-differential equations associated to our fractional calculus. We then discuss a fundamental problem concerning the Schrödinger equation in the framework of Nottale's scale relativity theory.

1. Introduction

In a previous paper [3], we have introduced, following a previous work of Kolvankar and Gangal [9], a new fractional calculus, which allows us to perform local analysis of non-differentiable functions. We have called α-derivative this new notion, which can be seen as a local version of the classical Riemann–Liouville derivative.

Many properties of the α-derivatives are given in [3], and we refer to this article for more details about this subject.

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In this article, we study fractional differential equations associated to the α-derivative. Such kind of equations appears in many problems. In particular, we have find a fractional differential equation related to the classical Schrödinger equation [4], by studying Nottale's approach to quantum mechanics via a fractal space–time [10].

This paper is organized as follow:

In Section 2, we recall some basic results on α-differentiability, $0 < \alpha < 1$, introduced in [3]. In [3] a generalized Taylor expansion theorem is obtain for α-differentiable function. The proof use the fact that $I^{\alpha}_{a,\sigma} \circ D^{\alpha}_{a,\sigma}[f(x) - f(a)](x) = f(x) - f(a)$, for $\sigma = \pm$, which is not true for an arbitrary function. We take the opportunity to give a complete proof of this result in Appendix A.

Section 3 introduces fractional differential equations associated to α-differentiation, and study some of their properties. Section 4 is devoted to linear fractional differential equations. Section 5 contains fundamental results about fractional differential equations of the form $\mathrm{d}^{\alpha} f(t) = a(t) + \mathrm{i}b(t)$, where $a(t)$ and $b(t)$ are continuous real valued functions, and d^{α} is the α-derivative. We prove many negative results. In particular, we prove that the fractional differential equations $\mathrm{d}^{\alpha} f(t) = a(t)$ or $\mathrm{d}^{\alpha} f(t) = \mathrm{i}a(t)$, $0 < \alpha < 1$, where $a(t)$ is a continuous real valued function, do not possess solutions.

In Section 6 we discuss the derivation of the Schrödinger equation in the scale relativity setting of Nottale [10]. We first gives basic results about the scale calculus introduced in [4]. We then define the *scale quantization procedure* of Newtonian mechanics developed in [4] following Nottale's approach [10] and state the *scale relativity principle*. We recall that the quantized analogue of the classical Newton equation of dynamics is a Schrödinger equation, as long as, there exists a non-trivial solution to the fractional differential equation $\mathrm{d}^{1/2} f(t) = \mathrm{i}C$, or $\mathrm{d}^{1/2} f(t) = C$, $C > 0$, $0 < \alpha < 1$, where $f(t)$ is the function describing the motion of the free particle. Using results from Section 5, we prove that we cannot make this assumption. As a consequence, we are lead to several new assumptions which we discuss in this paper.

2. Local fractional calculus

We refer to our paper [3] for more details and results about the notion of α-differentiation.

2.1. Riemann–Liouville differentiability

Let f be a continuous function on $[a, b]$. For all $x \in [a, b]$, we define the left (resp. right) Riemann–Liouville integral at point x by

$$I^{\alpha}_{a,-}(f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t,$$
$$I^{\alpha}_{b,+}(f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,\mathrm{d}t,$$

respectively.

The left (resp. right) Riemann–Liouville derivative at x is given by

$$D^{\alpha}_{a,-}(f)(x) = \frac{\mathrm{d}I^{1-\alpha}_{a,-}(f)(x)}{\mathrm{d}x},$$
$$D^{\alpha}_{b,+}(f)(x) = \frac{\mathrm{d}I^{1-\alpha}_{b,+}(f)(x)}{\mathrm{d}x}.$$

Definition 1. We say that the function f admits a derivative of order $0<\alpha<1$ (Riemann–Liouville) at $x\in[a,b]$ by below (resp. above) if $D^{\alpha}_{a,-}(f)(x)$ exists (resp. if $D^{\alpha}_{b,+}(f)(x)$ exists).

Of course, we obtain different values of the Riemann–Liouville derivative for different values of the parameter a (resp. b). Moreover, the derivative of a constant $C\in\mathbb{R}$ is not equal to zero. Indeed, we have

$$D^{\alpha}_{a,-}(C)(x) = \frac{C}{\Gamma(1-\alpha)}\frac{1}{(x-a)^{\alpha}}.$$

These two remarks give rise to great difficulties in the geometric interpretation of the Riemann–Liouville derivative. In particular, there is no relationship between the local geometry of the graph of f and its derivative, despite recent progress [2].

Definition 2. Let f be a continuous function on $[a,b]$, we call right (resp. left) local fractional derivative of f at $y\in[a,b]$ the following quantity

$$\frac{\mathrm{d}^{\alpha}_{\sigma}f}{\mathrm{d}y^{\alpha}}(y) = \lim_{x\to y^{\sigma}} D^{\alpha}_{y,-\sigma}[\sigma(f-f(y))](x), \qquad (1)$$

for $\sigma=\pm$ respectively.

We have the following obvious properties:

(I) (gluing) if f is differentiable at x, we have

$$\lim_{\alpha\to 1}\frac{\mathrm{d}^{\alpha}_{\sigma}f}{\mathrm{d}x^{\alpha}}(x) = f'(x), \quad \sigma=\pm.$$

(II) We have $\frac{\mathrm{d}^{\alpha}_{\pm}(C)}{\mathrm{d}x^{\alpha}} = 0$ for all $C\in\mathbb{R}$ and $\sigma=\pm$.

In [3], we obtain the following simplified equivalent definition of local fractional derivatives:

Theorem 1. *The (right or left) local fractional derivative of f, $d_\sigma^\alpha f(x)$ is equal to*

$$\frac{d_\sigma^\alpha f}{dx^\alpha}(x) = \Gamma(1+\alpha) \lim_{y \to x^\sigma} \frac{\sigma(f(y) - f(x))}{|\, y - x\,|^\alpha}. \tag{2}$$

The proof is based on the following generalized Taylor expansion theorem proved in [3]:

Theorem 2. *Let $0 < \alpha < 1$ and f be a continuous function, such that $d_\sigma^\alpha f(y)$ exists, $\sigma = \pm$. Then, we have*

$$f(x) = f(y) + \sigma \frac{1}{\Gamma(1+\alpha)} \frac{d_\sigma^\alpha f}{dx^\alpha}(y)[\sigma(x-y)]^\alpha + R_\sigma(x,y), \tag{3}$$

where $\lim_{x \to y^\sigma} \frac{R_\sigma(x,y)}{(\sigma(x-y))^\alpha} = 0$.

The proof of this theorem use the fact that the composition of the Riemann–Liouville fractional integral and the fractional derivative of $\Delta_a f(x) = f(x) - f(a)$, where a is the parameter, is equal to $\Delta_a f(x)$, which is not true in general. This result is implicit in [3]. As a consequence, we provide a complete statement as well as a proof in Appendix A.

Remark 1. The Riemann–Liouville fractional derivative is not a *derivation* [1] on the set of continuous functions. On the contrary, the differential operators $d_\sigma^\alpha/dt^\alpha$, $\sigma = \pm$, is a derivation and can be considered as a solution to the following problem:

Problem. Can we find non-trivial derivations on the set of non-differentiable functions?

By *non-trivial* we understand the two following conditions:

Let f be a continuous real valued function.

(i) The Hölder regularity of f can be read on the derivative;
(ii) One can recover the local geometry of the graph of f from the data of its derivative.

[1] We recall that an operator D acting on an algebra A is a derivation if it satisfies the Leibniz relation $D(xy) = Dx \cdot y + x \cdot Dy$ for all $x, y \in A$.

Conditions (i) and (ii) are satisfied by the (left and right) local fractional derivative. The maximal Hölder regularity of a given function is recover as the maximal order of local differentiation (see [4] for more details).

Moreover, using the local fractional derivative, we can obtain a local approximation of the graph of f via the generalized Taylor expansion theorem.

2.2. Non-differentiability and α-derivability

Let $f(t)$ be a continuous function on $[a,b]$. We remark that $\mathrm{d}^{\alpha}_{+}f(x) \neq \mathrm{d}^{\alpha}_{-}f(x)$ in general. In the differentiable case, we have (by (ii), Section 2.1), $\mathrm{d}^{1}_{+}f(x) = \mathrm{d}^{1}_{-}f(x)$. In other words, when $\alpha = 1$, the non-differentiability of a function is characterized by the existence of right and left local fractional derivatives, which carry different information on the local behaviour of the function. It is then necessary to introduce a new notion which takes into account these two data.

Definition 3. Let $f(t)$ be a continuous function on $[a,b]$ such that $\mathrm{d}^{\alpha}_{\sigma}f(y)$ exists for $\sigma = \pm$ and $y \in [a,b]$. We define the α-derivative of f at y, and we denote $\mathrm{d}^{\alpha}f/\mathrm{d}t^{\alpha}(y)$, the quantity

$$\frac{\mathrm{d}^{\alpha}f}{\mathrm{d}t^{\alpha}}(y) = \frac{1}{2}\left(\frac{\mathrm{d}^{\alpha}_{+}f}{\mathrm{d}t^{\alpha}}(y) + \frac{\mathrm{d}^{\alpha}_{-}f}{\mathrm{d}t^{\alpha}}(y)\right) + \mathrm{i}\frac{1}{2}\left(\frac{\mathrm{d}^{\alpha}_{+}f}{\mathrm{d}t^{\alpha}}(y) - \frac{\mathrm{d}^{\alpha}_{-}f}{\mathrm{d}t^{\alpha}}(y)\right), \tag{4}$$

where $i^2 = -1$.

In the following, we will use the notation $\mathrm{d}^{\alpha}f(y)$ instead of $\mathrm{d}^{\alpha}f/\mathrm{d}t^{\alpha}$ for shortness.

When f is differentiable, we have $\mathrm{d}^{1}f(y) = f'(y)$. If f is 1-differentiable, the non-differentiability is equivalent to the existence of an imaginary part for the 1-derivative.

Definition 4. A function f is said α-differentiable if the α-derivative exists at all points.

We denote by $\mathscr{C}^{\alpha}$ the set of α-differentiable functions.

3. Fractional differential equations

In the following, we denote by $\mathscr{H}^{\alpha}$, the set of continuous functions satisfying a Hölder condition $\mid f(x) - f(y) \mid < c \mid x - y \mid^{\alpha}$, $c > 0$.

3.1. Definitions and notations

We begin by a general remark about the terminology that we use.

Let D be a given differential operator on an algebra A. A *differential equation* on A is an equation of the form $\mathrm{D} \cdot x = f(x)$ for $x \in A$, and $f : A \to A$ a fixed map. This justify the following terminology of *fractional differential equations*:

Definition 5. A *fractional differential equation* of order α, $0 < \alpha < 1$, is an equation of the form

$$\frac{\mathrm{d}^\alpha y}{\mathrm{d}t^\alpha} = f(y, t), \tag{5}$$

where $y : \mathbb{R} \to \mathbb{R}^n$ is an α-differentiable function in the variable $t \in \mathbb{R}$, and $f(y, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}^n$ is a complex valued function.

A *Cauchy data* for (5) is an initial condition

$$y(t_0) = y_0, \tag{6}$$

where $y_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$.

3.2. The trivial case

We study the *trivial* case, i.e. fractal differential equations of the form

$$\frac{\mathrm{d}^\alpha y}{\mathrm{d}t^\alpha} = 0. \tag{7}$$

We first state the following trivial result:

Lemma 1. *Let $0 < \alpha < 1$ be given. For all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f \in \mathscr{H}^\gamma$, where $0 < \alpha < \gamma$, we have $\mathrm{d}^\alpha f / \mathrm{d}t^\alpha = 0$.*

We deduce the following important result:

Theorem 3. *The Cauchy problem $\mathrm{d}^\alpha y / \mathrm{d}t^\alpha = 0$, $y : \mathbb{R} \to \mathbb{R}$, with $y(t_0) = y_0$, admits an infinite set of solutions.*

Proof. The set of Hölderian functions $\mathscr{H}^\gamma$, $\gamma > \alpha$, belongs to the set of solutions by Lemma 1. Moreover, the Cauchy data does not allow us to fix a particular solution of this set. $\square$

A function satisfying $d^{\alpha}y/dt^{\alpha} = 0$ is called a *fixed function* [2] of order $0 < \alpha < 1$ in the following.

We denote by $\mathscr{F}^{\alpha}$ the set of fixed functions of order $0 < \alpha < 1$.

Remark 2. The characterization of the set $\mathscr{F}^{\alpha}$ is a difficult problem. It is related to our study of Section 5.

3.3. The general case

Using Theorem 3, we easily obtain the following result:

Theorem 4. *If a Cauchy problem* $d^{\alpha}y/dt^{\alpha} = f(t)$, $y(t_0) = y_0$ *admits a solution, then it admits an infinity of solutions.*

Proof. Let $y(t; t_0, y_0)$ be a solution, then $\tilde{y}(t; t_0, y_0) = y(t; t_0, y_0) + g(t; t_0, y_0)$, where g is a fixed function of order α such that $g(t_0) = 0$, is again a solution. As $\mathscr{F}^{\alpha}$ is an infinite set, this concludes the proof. □

This situation is very different from usual results for ordinary differential equations where we have unicity of solutions by fixing a Cauchy data. We can recover unicity in our case by introducing a notion of α-equivalence.

We first remark that two solutions of a Cauchy problem differs by a fixed function.

Definition 6. Let f and g be two continuous functions. We say that f is α-equivalent to g, and we denote $f \overset{\alpha}{\sim} g$ if and only if $f - g \in \mathscr{F}^{\alpha}$.

We easily prove that $\overset{\alpha}{\sim}$ is an equivalence relation.

Theorem 5. *Any Cauchy problem* $d^{\alpha}y/dt^{\alpha} = f(t)$, $y(t_0) = y_0$, *admits a unique solution in* $\mathscr{C}^{\alpha}/\overset{\alpha}{\sim}$.

Proof. Let $y_1(t)$ and $y_2(t)$ be two solutions. We have $y(t) = y_1(t) - y_2(t) \in \mathscr{F}^{\alpha}$. We deduce that $y_1 \overset{\alpha}{\sim} y_2$. This concludes the proof. □

We also have the following regularity result:

Theorem 6. *Any solution of a fractional differential equation of order* α *is a function belonging to* $\mathscr{C}^{\alpha}$.

[2] This terminology comes from the analogy existing between fixed functions and fixed points for ordinary differential equations, from the point of view of definitions.

4. Linear fractional differential equations

We discuss basic properties of linear fractional differential equations.

Definition 7. A linear fractional differential equation of order α, $0 < \alpha \leqslant 1$, is a fractional differential equation of the form

$$\frac{d^\alpha y}{dt^\alpha} = a(t)y + B(t), \tag{8}$$

where $a(t)$ and $B(t)$ are complex valued functions.

The homogeneous part of (8) is

$$\frac{d^\alpha y}{dt^\alpha} = a(t)y. \tag{9}$$

Lemma 2. *Let y_1 and y_2 be two solutions of* (9). *Then*, $Y = fy_1 + gy_2$, *where f and g belong to $\mathscr{F}^\alpha$, is also a solution of* (9).

Proof. As f and g belong to $\mathscr{F}^\alpha$, we have $d^\alpha f = d^\alpha g = 0$. We then obtain $d^\alpha(fy_1 + gy_2) = f\,d^\alpha y_1 + y_1\,d^\alpha f + g\,d^\alpha y_2 + y_2\,d^\alpha g$. We deduce that $d^\alpha(Y) = fa(t)y_1 + ga(t)y_2 = a(t)Y$, which concludes the proof. □

Lemma 3. *The solutions of a linear fractional differential equation* $d^\alpha y = a(t)y$ *are given by*

$$y(t) = k(t)\,e^{A(t)}, \tag{10}$$

where $k(t) \in \mathscr{F}^\alpha$ *and* $d^\alpha A(t) = a(t)$.

Proof. We have $d^\alpha(\log(y)) = (1/y)\,d^\alpha y$. We deduce that $d^\alpha(\log(y)) = a(t)$. Let $A(t)$ be such that $d^\alpha A(t) = a(t)$, then $\log(y) = A(t) + v(t)$, where $v \in \mathscr{F}^\alpha$. Hence, we have $y(t) = k(t)\,e^{A(t)}$ with $k(t) \in \mathscr{F}^\alpha$. □

We also have a construction of the solutions of a non-homogeneous fractional equation using a particular solution, and the general form of a solution to the homogeneous equation:

Lemma 4. *Let y_p be a particular* (*non-trivial*) *solution of* (8). *A function y is a solution of* (8) *if and only if it can take the form*

$$y = y_h + y_p, \tag{11}$$

where y_h is a solution of the associated homogeneous equation.

Proof. We have $d^\alpha y_h = a(t)y_h$ and $d^\alpha y_p = a(t)y_p + B(t)$. Hence, we obtain $d^\alpha(y_h + y_p) = a(t)(y_h + y_p) + B(t)$.

Let y be a solution of (8). Then, we have

$$d^\alpha(y - y_p) = a(t)y + B(t) - a(t)y_p - B(t) = a(t)(y - y_p). \tag{12}$$

The function $y - y_p$ is a solution of the homogeneous equation. This concludes the proof. □

In order to compute a particular solution, one must have a method similar to the classical "variation of constant" for ordinary differential equations. Precisely, the general form of a solution for the linear equation is $y(t) = k(t)\exp A(t)$, where $k(t) \in \mathscr{F}^\alpha$. We look for a solution of the form $y(t) = c(t)\exp A(t)$ with $c(t) \in \mathscr{C}^\alpha$. A simple computation gives

$$d^\alpha c(t) = B(t)\exp(-A(t)). \tag{13}$$

By solving this fractional differential equation, simpler than the initial one, we obtain a particular solution.

Theorem 7. *Let y_1 and y_2 be two solutions of the fractional equations*

$$d^\alpha y = a(t)y + B_1(t) \quad \textit{and} \quad d^\alpha y = a(t)y + B_2(t), \tag{14}$$

respectively.

Then, $y_1 + y_2$ is a solution of the fractional differential equation

$$d^\alpha y = a(t)y + (B_1(t) + B_2(t)). \tag{15}$$

Proof. This is a simple computation. □

4.1. Fractional differential equations of order $\alpha + n$, $n \in \mathbb{N}$, $0 < \alpha < 1$

Let f be a differentiable function of class C^n. We say that f is of class $C^{n+\alpha}$, $0 < \alpha < 1$, if

$$d_\sigma^{\alpha+n} f(x) = d_\sigma^\alpha(f^{(n)})(x), \quad \sigma = \pm, \tag{16}$$

exists for all $x \in \mathbb{R}$, where $f^{(n)}$ is the n-term derivative of f.

In [3], we prove the following Taylor's expansion theorem:

Theorem 8. *Let* $0 < \alpha < 1$, $f \in C^{n+\alpha}$, *then*

$$f(x) = f(y) + \sum_{i=1}^{n} \frac{f^{(i)}(y)}{\Gamma(i+1)}(x-y)^i + \sigma \frac{d_\sigma^\alpha f^{(n+1)}(y)}{\Gamma(n+\alpha+1)}[\sigma(x-y)]^{n+\alpha} + R_\sigma(x,y), \tag{17}$$

with $\lim_{x \to y^\sigma} \frac{R_\sigma(x,y)}{(\sigma(x-y))^{\alpha+n}} = 0$, $\sigma = \pm$.

We then are lead to the following notion of fractional differential equation of order $\alpha + n$, $0 < \alpha < 1$:

Definition 8. A fractional differential equation of order $\alpha + n$, is an equation of the form

$$\frac{d^{\alpha+n} y}{dt^{\alpha+n}} = f(t, y, y', \ldots, y^{(n-1)}), \tag{18}$$

where f is a function defined on an open set of $\mathbb{R} \times \mathbb{R}^n$.

The classical idea is to study fractional differential equations of order $\alpha + n$, $n \in \mathbb{N}$ by fractional differential equation of order α, but in a bigger space.

Let

$$z_k = \frac{d^k z}{dt^k}, \quad k = 1, \ldots, n. \tag{19}$$

The fractional differential equation of order $\alpha + n$ (18) is equivalent to

$$\begin{aligned} \frac{d^\alpha z_n}{dt^\alpha} &= f(t, y, z_1, \ldots, z_{n-1}), \\ \frac{dz_{n-1}}{dt} &= z_n, \\ &\vdots \\ \frac{dy}{dt} &= z_1. \end{aligned} \tag{20}$$

We have the following result:

Theorem 9. *Solutions* y *of a fractional differential equation* (18) *of order* $\alpha + n$ *are given by the solutions of the system* (20).

5. About fractal differential equations of the form $d^\alpha y = a(t) + ib(t)$

In this section we investigate properties of fractional differential equations of the form

$$\frac{\mathrm{d}^\alpha y(t)}{\mathrm{d}t^\alpha} = a(t) + \mathrm{i}b(t), \tag{21}$$

where $a(t)$ and $b(t)$ are two real valued continuous functions defined on $\mathbb{R}$.

5.1. Constant fractional differential equations

Let $[a,b]$ be a compact interval of $\mathbb{R}$. If f is continuous, then, there exists at least two points $u, U \in [a,b]$ such that

$$f(u) \leqslant f(x) \leqslant f(U) \quad \forall x \in [a,b]. \tag{22}$$

With these notations, we prove the following lemma:

Lemma 5. *If* $\mathrm{d}^\alpha_+ f(x) = 1\ \forall x \in [a,b]$, *then* $U = b$.

Proof. If $U \in]a,b[$, then U is a local maxima and we must have $\mathrm{d}^\alpha_+ f(U) < 0$. This is impossible by assumption.

If $U = a$, then U is such that $f(U) \geqslant f(x)$ for all $x \in [a,b]$. We deduce that $\mathrm{d}^\alpha_+ f(a) < 0$, which is again a contradiction.

If $U = b$, the only constraint is on the left derivative of b, and we can have $\mathrm{d}^\alpha_+ f(b) = 1$. □

Lemma 6. *If* $\mathrm{d}^\alpha_- f(x) = -1$ *for all* $x \in [a,b]$, *then* $U = a$.

Proof. If $U \in]a,b[$, then U must be a local maximum and we obtain a contradiction. If $U = b$, as we have $f(U) \geqslant f(x)\ \ \forall x \in [a,b]$, we deduce $\mathrm{d}^\alpha_- f(U) \geqslant 0$, which is impossible. The only possible case is then $U = a$. □

We deduce from these two lemmas:

Theorem 10. *The fractional differential equation* $\mathrm{d}^\alpha f(x) = i$ *has no solutions for* $0 < \alpha \leqslant 1$.

Proof. As $\mathrm{d}^\alpha f = i$, we have

$$\mathrm{d}^\alpha_+ f(x) = 1 \quad \text{and} \quad \mathrm{d}^\alpha_- f(x) = -1 \quad \forall x \in \mathbb{R}. \tag{23}$$

Let $[a,b]$ be an arbitrary closed interval of $\mathbb{R}$. By Lemmas 6 and 5, we deduce that $U = a$ and $U = b$. Then, f is a constant function on $[a,b]$.

If $0 < \alpha < 1$, then f is non-differentiable. As f is a constant function, it is differentiable, in contradiction with the assumption $0 < \alpha < 1$.

If $\alpha = 1$, we obtain again a contradiction. Indeed, as f is a constant function (then differentiable), we have $\mathrm{d}^1_+ f(x) = d^1_- f(x)$ in contradiction with (23). This concludes the proof. □

We also prove:

Theorem 11. *The fractional differential equation* $d^\alpha f = 1$ *has no solutions for* $0 < \alpha < 1$.

Proof. The fractional differential equation $d^\alpha f = 1$ implies that

$$d^\alpha_+ f(x) = d^\alpha_- f(x) = 1 \quad \forall x \in \mathbb{R}. \tag{24}$$

We now prove that f is injective. Let x_1 and x_2 be two points of $[a,b]$ such that $f(x_1) = f(x_2)$. Then, by the generalized Rolle theorem (see [4]), there exists a point $c \in [x_1, x_2]$, such that $d^\alpha_+ f(c)\, d^\alpha_- f(c) \leqslant 0$. This is impossible by assumption. Then, f is injective.

As f is injective and continuous, we know that f is strictly monotone [7, Lemma 3.8, p. 207]. By Lebesques's theorem [8, p. 319], this function is almost everywhere differentiable. As $0 < \alpha < 1$, this is impossible. This concludes the proof. □

Theorems 10 and 11 allow us to prove:

Theorem 12. *Fractional differential equations of the form* $d^\alpha f = a(t)$ *and* $d^\alpha f = ia(t)$ *where* $a(t)$ *belongs to* $\mathscr{C}^\gamma$, $\gamma > \alpha$, *and such that* $a(t) \neq 0$, *have no solutions.*

Proof. Let f be a solution of $d^\alpha f = a(t)$. Then, the function $f(t)/a(t)$, which is a well-defined function because $a(t) \neq 0$, is a non-trivial solution of $d^\alpha y = 1$, which is impossible by Theorem 11.

Indeed, we have $d^\alpha(f(t)/a(t)) = (d^\alpha f \cdot a(t) - f(t) \cdot d^\alpha a)/a^2(t)$. As $a(t) \in \mathscr{C}^\gamma$, and $\gamma > \alpha$, we deduce that $d^\alpha a = 0$. Hence, we obtain $d^\alpha(f/a) = 1$.

In a similar way, we prove that $d^\alpha f = ia(t)$ has no solutions. □

Along the same lines, we prove:

Lemma 7. *The fractional differential equation* $d^\alpha f(x) = 1 + i$ *has no solutions if* $0 < \alpha < 1$.

Proof. As $d^\alpha f = 1 + i$, we deduce that

$$d^\alpha_+ f(x) = 2 \quad \text{and} \quad d^\alpha_- f(x) = 0 \qquad \forall x \in \mathbb{R}. \tag{25}$$

Let $[a,b]$ be a closed interval of $\mathbb{R}$. As $d^\alpha_+ f(x) = 2$ on $[a,b]$, we have $U = b$. Let $u \in]a,b[$ be a global minimum of f on $[a,b]$, then $f(x) \geqslant f(u)$ $\forall x \in [a,b]$. Hence, there exists $x_0 \in [a,u[$ and $\delta > 0$ such that $f(x_0) \geqslant f(x)$ $\forall x \in [x_0, x_0 + \delta[$. This implies $d^\alpha_+ f(x_0) \leqslant 0$, which is impossible by assumption. Then, we have $u = a$. We prove, along the same line as the proof of Theorem 11, that f is strictly increasing, and as a consequence, derivable almost everywhere by Lebesque theorem. As $0 < \alpha < 1$, we have a contradiction. This concludes the proof. □

We deduce from Lemma 7:

Theorem 13. *Fractional differential equations of the form* $\mathrm{d}^\alpha f(t) = a(t)(1 + \mathrm{i})$, *with* $a(t) \neq 0$ $\forall t \in \mathbb{R}$ *and* $a(t) \in \mathscr{F}^\alpha$, *have no solutions for* $0 < \alpha < 1$.

Proof. Let f be a solution of $\mathrm{d}^\alpha f(t) = a(t)(1 + \mathrm{i})$, then $\mathrm{d}^\alpha (f/a) = (1/a)\,\mathrm{d}^\alpha f + f\,\mathrm{d}^\alpha (1/a)$. As $a \in \mathscr{F}^\alpha$, we have $\mathrm{d}^\alpha(1/a) = 0$. Hence, we obtain $\mathrm{d}^\alpha(f/a) = 1 + \mathrm{i}$. By Lemma 7, this equation has no solutions. This concludes the proof. □

5.2. A conjecture

All our results leads to the following conjecture:

Conjecture. *Fractional differential equations of the form* $\mathrm{d}^\alpha f(t) = a(t) + \mathrm{i}b(t)$, $0 < \alpha < 1$, *where* $a(t)$ *and* $b(t)$ *are continuous functions, have no solutions.*

Remark 3. The fractional differential equations that we have up to now considered are strongly constrained. Indeed, we assume that the order of fractional differentiation is constant, which is a strong assumption on the Hölderian behaviour of the possible solution (this means that the solution has a uniform Hölder exponent, using results from [3]).

We can generalize our definition, by assuming a non-uniform Hölderian behaviour. We are lead to consider fractional equations of the form

$$\mathrm{d}^{\alpha(t)} f = a(t) + \mathrm{i}b(t), \tag{26}$$

where $\alpha(t)$ is a continuous (or not) real valued function, such that $0 < \alpha(t) < 1$.

We refer to [14] for a first approach to fractional differentiation of variable fractional order.

6. About the Schrödinger equation

The Schrödinger equation control the dynamical behaviour of quantum particles. It can be obtained by the quantum mechanics formalism. Following an idea of Nottale [10], we have proved [4], introducing a new kind of differential calculus, called the scale calculus, that the Schrödinger equation is the classical Newton equation of motion but for a free particle on a fractal space–time.

6.1. Reminder about the scale calculus

The scale calculus generalize the classical differential calculus by introducing a notion of minimal resolution.

Let f be a continuous real valued functions. We define in [4] a real number $\tau(f)$ called the *minimal resolution of* f, such that $\tau(f) = 0$ for an everywhere differentiable function and $\tau(f) > 0$ for an everywhere non-differentiable function.

We then introduce *left and right quantum derivatives* of f at point t as

$$\frac{\Box_\sigma f}{\Box t}(t) = \lim_{h\to\tau(f)} \sigma \frac{f(t) - f(t+\sigma h)}{h}, \quad \sigma = \pm. \tag{27}$$

We note that when f is differentiable, then $\tau(f) = 0$, and we recover the classical left and right derivatives.

The *scale derivative* of f at point t combines these two quantities in such a way that we recover the classical derivative when f is differentiable:

$$\frac{\Box f}{\Box t}(t) = \frac{1}{2}\left[\frac{\Box_+ f}{\Box t} + \frac{\Box_- f}{\Box t}\right] - \mathrm{i}\left[\frac{\Box_+ f}{\Box t} - \frac{\Box_- f}{\Box t}\right]. \tag{28}$$

We can extend this definition to complex valued functions as follow.

Let f be a complex valued continuous function. We denote by $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ the real and imaginary part of f which are real valued continuous functions. The scale derivative of f is defined as

$$\frac{\Box f}{\Box t} = \frac{\Box\,\mathrm{Re}(f)}{\Box t} + \mathrm{i}\frac{\Box\,\mathrm{Im}(f)}{\Box t}. \tag{29}$$

Remark 4. The extension of the scale calculus to complex valued functions is not trivial (as in the case of local fractional calculus) as it mixes complex terms in a complex operator.

The main formula that we use in the following is (see [4] for a proof):

Let $X(t)$ be in $\mathscr{C}^{1/2}$, and $C(X,t)$ be a C^2 complex valued function. The scale derivative of $\mathscr{C}(t) = C(X(t), t)$, is given by

$$\frac{\Box\mathscr{C}}{\Box t} = \frac{\Box x}{\Box t}\frac{\partial\mathscr{C}}{\partial x} + \frac{1}{2}a(t)\frac{\partial^2\mathscr{C}}{\partial x^2} + \frac{\partial\mathscr{C}}{\partial t}, \tag{30}$$

where

$$a(t) = \left(\frac{(\mathrm{d}_+^{1/2} x(t))^2 - (\mathrm{d}_-^{1/2} x(t))^2}{2} \right) - \mathrm{i} \left(\frac{(\mathrm{d}_+^{1/2} x(t))^2 + (\mathrm{d}_-^{1/2} x(t))^2}{2} \right). \tag{31}$$

6.2. The scale quantized Newton equation and the Schrödinger equation

Classical mechanics is based on the Lagrangian formalism for the Newton equation of dynamics. Let $x(t)$ be the trajectory of a point-mass of mass m in a given potential $U(x,t)$. The classical Lagrangian associated to the dynamics of x is

$$L(x, v, t) = \frac{1}{2} m v^2 + U(x, t), \tag{32}$$

where v is the classical speed of x given by $v = \mathrm{d}x/\mathrm{d}t$.

We obtain the Newton equation of dynamics by writing the Euler–Lagrange equation associated to L:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial v} \right) = \frac{\partial L}{\partial x}. \tag{33}$$

The scale relativity principle developed by Nottale [10] implies that the Newton equation of dynamics is related to a quantum analogue by the following quantization procedure:

Scale quantization principle: The equation of dynamics keep the same form, but the trajectories belongs to $\mathscr{C}^{1/2}$ and the classical time derivative $\partial/\partial t$ is replaced by the scale derivative $\square/\square t$.

Remark 5. Our assumption that quantum trajectories belongs to $\mathscr{C}^{1/2}$ is based on Feynman–Hibbs [6] characterization of typical paths of quantum mechanics (see Section 6.4). Moreover, the Heisenberg uncertainty relations which implies that the Hausdorff dimension of a quantum mechanical path is 2 (see [1]), is consistent with this assumption.

As a consequence, if we denote by $Q(\cdot)$ the map associating to each variable or differential operator its quantum counterpart, we have the following rules:

(i) $x \in C^1$, $X = Q(x) \in \mathscr{C}^{1/2}$,
(ii) $v \in \mathbb{R}$, $v = \mathrm{d}x/\mathrm{d}t$, and $V = Q(v) \in \mathbb{C}$, $V = \square X/\square t$,
(iii) $t \in \mathbb{R}$, $t = Q(t)$.

Remark 6. The last condition meaning that we have not assume that the time variable is itself a fractal variable.

As a consequence, the classical Lagrangian $L(x, v, t)$ has the following quantum analogue:

$$\mathscr{L}(X, V, t) = \frac{1}{2} m V^2 + U(x, t). \tag{34}$$

Of course, by definition of V, the quantum Lagrangian $\mathscr{L}$ is a *complex* valued function.

The quantized Newton equation of dynamics is given by:

$$\frac{1}{2} m \frac{\Box V}{\Box t} = \frac{\partial U}{\partial X}. \tag{35}$$

A classical notion associated to Lagrangian mechanics is the *action*, denoted by $A(x, t)$, and associated to v by the formula

$$mv = \frac{\partial A}{\partial x}. \tag{36}$$

We denote by $\mathscr{A}(X, t)$ the quantum analogue of A. We then have

$$mV = \frac{\partial \mathscr{A}}{\partial X}. \tag{37}$$

A basic consequence of the non-differentiability of X being the complex nature of V, we introduce a complex valued function $\psi(X, t)$ defined by

$$\psi(X, t) = \exp\left(\frac{\mathrm{i}\mathscr{A}}{2m\gamma}\right), \tag{38}$$

where $\gamma \in \mathbb{R}$ is a normalization constant. The function ψ is the well-known *wave function*.

Remark 7. In quantum mechanics the wave function is introduced by hand. In the context of the scale relativity theory the wave function is only a reflection of the loss of differentiability of the space–time structure via the complex nature of the speed V.

Using the wave function, we can express the quantum speed V as

$$V = -\mathrm{i}2\gamma \frac{\partial \ln(\psi)}{\partial X}. \tag{39}$$

The quantized Newton equation of dynamics (35) written in term of ψ is given by:

$$2\mathrm{i}\gamma m \frac{\Box}{\Box t}\left(\frac{\partial}{\partial X}(\ln(\psi))\right) = \frac{\partial U}{\partial X}. \tag{40}$$

Using formula (30) for the complex valued function

$$\mathscr{C}(t) = \frac{\partial \ln(\psi)}{\partial X}(X(t), t), \tag{41}$$

we prove in [4] that the quantized Newton equation is equivalent to the following *generalized non-linear Schrödinger equation* [4, Lemma 6, Section 5.2]:

$$-\mathrm{i}2\gamma m\left(\mathrm{i}\gamma + \frac{a(t)}{2}\right)\left(\frac{\partial\psi}{\partial X}\right)^2\frac{1}{\psi^2} + \mathrm{i}2\gamma\frac{\partial \ln\psi}{\partial t} + \mathrm{i}\gamma a(t)\frac{\partial^2\psi}{\partial X^2}\frac{1}{\psi} = U(X,t) + \alpha(X), \tag{42}$$

where

$$a(t) = \left(\frac{(\mathrm{d}_+^{1/2}X(t))^2 - (\mathrm{d}_-^{1/2}X(t))^2}{2}\right) - \mathrm{i}\left(\frac{(\mathrm{d}_+^{1/2}X(t))^2 + (\mathrm{d}_-^{1/2}X(t))^2}{2}\right), \tag{43}$$

and $\alpha(X)$ is an arbitrary continuous function.

It is possible to obtain the *classical linear* Schrödinger equation by imposition a condition on the value of $a(t)$ [4, Corollary 1]:

Corollary 1. *If the function $X(t)$ is such that*

$$a(t) = -\mathrm{i}2\gamma, \tag{SC}$$

then Eq. (42) *takes the form*

$$\mathrm{i}2\gamma m\frac{\partial\psi}{\partial t} + 2\gamma^2 m\frac{\partial^2\psi}{\partial X^2} = (\Phi + \alpha(X))\psi. \tag{44}$$

We can always choose a solution of (44) *such that* $\alpha(X) = 0$. *In this case*, *when*

$$\gamma = \frac{\bar{h}}{2m}, \tag{45}$$

where $\bar{h}$ is the Planck constant, *we obtain the classical Schrödinger's equation*

$$\mathrm{i}\bar{h}\frac{\partial\psi}{\partial t} + \frac{\bar{h}^2}{2m}\frac{\partial^2\psi}{\partial X^2} = \Phi\psi. \tag{46}$$

We call condition (SC) the Schrödinger condition in the following.

6.3. The Schrödinger condition and fractional differential equations

From (43), we deduce that the Schrödinger condition (SC) is equivalent to

$$(\mathrm{d}_+^{1/2}X(t))^2 = (\mathrm{d}_-^{1/2}X(t))^2, \tag{47}$$

and

$$(\mathrm{d}_+^\alpha X(t))^2 = 2\gamma. \tag{48}$$

Hence, we are lead to the following problem:

Problem. Does there exist a continuous function satisfying

$$(\mathrm{d}_+^{1/2} X(t))^2 = (\mathrm{d}_-^{1/2} X(t))^2 = C, \tag{49}$$

where $C > 0$ is a constant.

Using results from the previous section, we prove the following theorem:

Theorem 14. *For all* $0 < \alpha < 1$, *the fractional equation*

$$(\mathrm{d}_+^\alpha X(t))^2 = (\mathrm{d}_-^\alpha X(t))^2 = b(t), \tag{50}$$

where $b(t) \geqslant 0$ *is a continuous function, has no solutions.*

Proof. This follows easily from Theorem 12. Indeed, we have only two cases to consider:

(i) $d_+^\alpha X(t) = \mathrm{d}_-^\alpha X(t) = \pm\sqrt{b(t)}$,
(ii) $d_+^\alpha X(t) = -\mathrm{d}_-^\alpha X(t) = \pm\sqrt{b(t)}$.

For (i), we obtain $\mathrm{d}^\alpha X(t) = \pm 2\sqrt{b(t)}$ and for (ii), we have $\mathrm{d}^\alpha X(t) = \pm \mathrm{i}2\sqrt{b(t)}$. By Theorem 12, as $\sqrt{b(t)}$ is again a continuous function, we have no solutions to (50). □

6.4. Feynman–Hibbs characterization of quantum paths and generalized Schrödinger equations

As a consequence of Theorem 14, we cannot assume that $a(t)$ satisfies the Schrödinger condition (SC). This means that, in the scale relativity point of view, the equation of motion of a free particle is a generalized form of the Schrödinger equation, like (49).

In the following, we precise the form of (49) by exploring the most general form which can be assumed for $a(t)$, compatible with standard results of quantum mechanics.

In [6], Feynman and Hibbs characterize typical paths of quantum mechanics. They prove that they are continuous everywhere non-differentiable curves for which a *quadratic velocity* can be defined. In our framework, the *Feynman–Hibbs characterization* of quantum trajectories is then defined by:

Feynman–Hibbs characterization: A function $X(t)$ is said to satisfy the Feynman–Hibbs condition if

$$\lim_{h\to 0^+}\frac{(X(t+h)-X(t))^2}{h}=\lim_{h\to 0^+}\frac{(X(t)-X(t-h))^2}{h}. \tag{FH}$$

The Feynman–Hibbs condition (FH) has important implications in our framework.

Lemma 8. *If $X\in\mathscr{C}^{1/2}$ and X satisfies* (FH) *then*

$$(\mathrm{d}_+^{1/2}X(t))^2=(\mathrm{d}_-^{1/2}X(t))^2, \tag{51}$$

and the most general form for the Schrödinger condition consistent with (FH) *is*

$$a(t)=-\mathrm{i}b(t,X(t)), \tag{GSC}$$

where $b(t,X)$ is a positive definite arbitrary function.

Proof. As $X(t)$ is 1/2-derivable, then by the generalized Taylor theorem [3], we have

$$\begin{aligned}(X(t+h)-X(t))^2&=(\mathrm{d}_+^{1/2}X(t))^2h+\mathrm{o}(h),\\ (X(t)-X(t-h))^2&=(\mathrm{d}_-^{1/2}X(t))^2h+\mathrm{o}(h).\end{aligned} \tag{52}$$

As X satisfies (FH), we deduce that $(\mathrm{d}_+^{1/2}X(t))^2=(\mathrm{d}_-^{1/2}X(t))^2$.

By definition of $a(t)$, we have under condition (FH)

$$a(t)=-\mathrm{i}(\mathrm{d}_+^{1/2}X(t))^2. \tag{53}$$

As a consequence, the most general form of the Schrödinger condition is $a(t)=-ib(t,X(t))$, where $b(t,X)$ is a positive definite function. This concludes the proof of the lemma. □

The generalized Schrödinger condition (GSC) implies that we must study the following fractional differential equations:

$$(\mathrm{d}_+^{1/2}X(t))^2=(\mathrm{d}_-^{1/2}X(t))^2=b(t,X(t)), \tag{54}$$

where $b(t,X)$ is a positive definite function.

Several cases must be considered.

6.4.1. Independence

If b is independent of X, i.e. $b(t,X)=b(t)$, we have the following possibilities:

(i) $b(t)$ is a continuous function;
(ii) $b(t)$ is discontinuous;
(iii) mixing of continuity and discontinuity;

Case (i) is impossible. Indeed, the Theorem 14 applies, and we have no solutions to the fractional differential equation (54).

Case (ii) cannot be avoid. However, we have strong constraints for the function $b(t)$ coming from physics.

Indeed, the Schrödinger equation is a well-established equation of physics for the dynamics of quantum particle in the non-relativistic case, in agreement with experimental results. As a consequence, the function $b(t)$ must satisfy the following *reality condition*:

$$\mid b(t) - \bar{h}/m \mid \leqslant \epsilon, \tag{R}$$

where $0 < \epsilon \ll 1$ is small parameter.

Indeed, even if the function $b(t)$ is discontinuous, we cannot allow large fluctuations with respect to the value $\bar{h}/m$ leading to the classical Schrödinger equation in Eq. (42).

Then, we are lead to consider fractional differential equations of the form

$$(\mathrm{d}_+^{1/2}X(t))^2 = (\mathrm{d}_-^{1/2}X(t))^2 = \bar{h}/m + \epsilon P(t), \tag{55}$$

where $0 < \epsilon \ll 1$ is a small parameter and $P(t)$ is a discontinuous function such that $\bar{h}/m + \epsilon P(t) \geqslant 0$.

For the same reasons as case (ii), (iii) leads to the study of fractional differential equations of the form (55).

Assuming that the fractional differential Eq. (55) has a solution, we obtain the following form for the generalized Schrödinger equation, compatible with the Feynman–Hibbs condition (FH) and the reality condition (R):

$$\mathrm{i}\bar{h}\frac{\partial\psi}{\partial t} + \frac{\bar{h}^2}{2m}\frac{\partial^2\psi}{\partial X^2} + \epsilon\mathscr{P}(t,\psi) = U\psi, \tag{56}$$

where the perturbation $\mathscr{P}$ is given by

$$\mathscr{P}(t,\psi) = \frac{\bar{h}}{2m}P(t)\left[\frac{\partial^2\psi}{\partial X^2} - m\left(\frac{\partial\psi}{\partial X}\right)^2\frac{1}{\psi}\right]. \tag{57}$$

Many works deal with generalization of the classical Schrödinger equation by adding non-linear terms in order to solve some specific problems of quantum mechanics such as the collapse of the wave function (see [11,13] for example) or the cat paradox. However, the non-linear terms are in general ad hoc and justified a posterior.

In our case, the non-linear Schrödinger Eq. (56) has a specific non-linear term (57) which is fixed by the Feynman–Hibbs condition, the reality condition and the scale quantization procedure coming from the scale relativity theory of Nottale [10].

It will be interesting to exhibit new phenomenon associated to this non-linear term.

6.4.2. Dependence

If b is dependent of $X(t)$, we cannot conclude. As in the previous paragraph, we must satisfy the reality condition. As a consequence, we are lead to consider a generalized non-linear Schrödinger equation of the form (56), with a non-linear term given by (57) where $P(t)$ is replaced by a composed function $P(t, \psi(t))$.

6.5. Further generalizations

Up to now, we have considered fractional differential equations with a constant order of differentiation. From the physical view-point, this comes from the Feynman–Hibbs condition (FH) which implies that this order is 1/2. However, this is an *averaged* value. As a consequence, a better characterization of quantum paths is the following:

Absolute Feynman–Hibbs characterization: A function $X(t)$ is said to satisfy the absolute Feynman–Hibbs condition if

$$\lim_{h\to 0} \frac{(f(t+h)-f(t))^{2+\epsilon p(t)}}{h} \text{ exists}, \tag{58}$$

where $0 < \epsilon \ll 1$ is a small parameter and $p(t)$ is continuous function such that

$$p_\epsilon(t) = \frac{1}{2}\int_{t-\epsilon}^{t+\epsilon} p(s)\,\mathrm{d}s = 0. \tag{59}$$

Remark 8. This assumption is consistent with the *special scale relativity theory* (see [5,10]). A prediction of this theory is that quantum mechanical paths must have a variable *fractal dimension* (see [15]). As a consequence, we have ([4, Theorem 2.4] and [15]) a variable Hölderian exponent which gives a variable order of differentiation.

In this case, the non-linear Schrödinger equation is more complicated and we are lead to consider fractional differential equations with a variable order of differentiation (see [14] for a first approach). It will be interesting to write a detailed analysis of this case.

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Appendix A. A technical lemma

In [3], we have implicitly used the fact that

$$I^{\alpha}_{a,\sigma} \circ D^{\alpha}_{a,\sigma}[f(x) - f(a)](x) = f(x) - f(a), \tag{A.1}$$

in the proof of the generalized Taylor theorem in [3]. This result is not true if we replace $f(x) - f(a)$ by an arbitrary function. However, in our case, a special phenomenon occurs.

Lemma 9. *Let f be a continuous function, $0 < \alpha < 1$ and $a \in \mathbb{R}$ be a given real number. We denote by $\Delta_a f(x)$ the difference $\Delta_a f(x) = f(x) - f(a)$. We have*

$$I^{\alpha}_{a,\sigma} \circ D^{\alpha}_{a,\sigma}[\Delta_a f](x) = \Delta_a f(x), \tag{A.2}$$

for $\sigma = \pm$.

Proof. If $F(z)$ is an arbitrary continuous function, then we have (see [12, p. 71, (2.113)]):

$$I^{\alpha}_{a,\sigma} \circ D^{\alpha}_{a,\sigma} F(x) = F(x) - [I^{1-\alpha}_{a,\sigma} F(x)]_{x=a} \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}. \tag{A.3}$$

Podlubny has proved (see [12, p. 75, (2.128) and (2.130), and Section 2.3.7]) that if $0 < \alpha < 1$, then the condition

$$[I^{1-\alpha}_{a,\sigma} F(x)]_{x=a} = 0 \tag{A.4}$$

is equivalent to

$$F(a) = 0. \tag{A.5}$$

In our case this is trivial because $\Delta_a f(a) = 0$. As a consequence, using (A.3) with $F(x) = \Delta_a f(x)$, we obtain

$$I^{\alpha}_{a,\sigma} \circ D^{\alpha}_{a,\sigma} \Delta_a f(x) = \Delta_a f(x). \qquad \square \tag{A.6}$$

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