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About Non-differentiable Functions

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We develop the notion of local fractional derivative introduced by Kolvankar and Gangal. It allows a fine study of the local structure of irregular (fractal) functions. Using this tool, we extend classical theorems of analysis (extrema, Rolle) to non-differentiable functions. In particular, we prove a generalized Taylor expansion theorem. We introduce a new derivative of real order and discuss its properties.

Key Words: fractional calculus; Riemann–Liouville fractional operators; irregular functions.

INTRODUCTION

Several natural phenomena lead to irregular ("fractal") objects. For example, typical paths of quantum mechanical particles are continuous but non-differentiable. Despite the ubiquity of non-differentiable structures in nature, we have few mathematical tools to deal with.

An idea is to generalize the notion of derivative in order to take into account non-differentiable functions. Many attempts already exists, in particular, the so-called *fractional derivative* of Riemann–Liouville, Liouville,

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Weyl, and Marchaud [1, 16]. They are all, more or less, based on a generalization of the Cauchy formula. Hence, there is no *geometric* idea supporting these generalizations, explaining the difficulties of using it in order to obtain information about the structure of non-differentiable objects. Moreover, fractional derivatives are all *non-local* on the contrary of the classical derivative. For example, the Riemann–Liouville derivative depends on a free parameter which relies on a global information on the function. The study of non-differentiable functions via these operators is then difficult.

In this article, we solve this problem by introducing a notion of (right or left) *local fractional derivative*, following a previous work of Kolwankar and Gangal [5]. It is defined, at a point y, by taking the Riemann–Liouville derivative of $f(x) - f(y)$ and by tending the free parameter toward y. This simple change induces many interesting properties.

First, contrary to Riemann–Liouville, the derivative of a constant function is zero. This allows us to generalize classical results of analysis (extrema, Rolle, Taylor) to the non-differentiable case.

Second, we have a clear geometrical meaning of the derivative: it gives the local Hölderian behavior of the function, and the critical order of derivation is equal to the Hölder exponent.

In order to reconstruct the local behaviour of a non-differentiable function, it is necessary to have the right and left local fractional derivative. We then introduce a new derivative, called the *α-derivative*, which summarizes all the information that we need to perform this local analysis. The α-derivative number is a complex number. When the function is differentiable the imaginary part disappears and we obtain the usual derivative.

We introduce the space of $\mathscr{C}^\alpha$ functions (which possess an *α-derivative*) and we study its properties.

1. LOCAL FRACTIONAL DERIVATIVE

1.1. *Riemann–Liouville Differentiability*

Let f be a continuous function on $[a, b]$. For all $x \in [a, b]$, we define the left (resp. right) Riemann–Liouville integral at point x by

$$I^{\alpha}_{a,-}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt,$$

$$I^{\alpha}_{b,+}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt,$$

respectively.

The left (resp. right) Riemann–Liouville derivative at x is given by

$$D^{\alpha}_{a,-}(f)(x) = \frac{dI^{1-\alpha}_{a,-}(f)(x)}{dx},$$

$$D^{\alpha}_{b,+}(f)(x) = \frac{dI^{1-\alpha}_{b,+}(f)(x)}{dx}.$$

DEFINITION 1.1. We say that the function f admits a derivative of order $0 < \alpha < 1$ (Riemann–Liouville) at $x \in [a, b]$ by below (resp. above) if $D^{\alpha}_{a,-}(f)(x)$ exists (resp. if $D^{\alpha}_{b,+}(f)(x)$ exists).

Of course, we obtain different values of the Riemann–Liouville derivative for different values of the parameter a (resp. b). Moreover, the derivative of a constant $C \in \mathbb{R}$ is not equal to zero. Indeed, we have

$$D^{\alpha}_{a,-}(C)(x) = \frac{C}{\Gamma(1-\alpha)} \frac{1}{(x-a)^{\alpha}}.$$

These two remarks give rise to great difficulties in the geometric interpretation of the Riemann–Liouville derivative [1]. In particular, there is no relationship between the local geometry of the graph of f and its derivative.

Properties of the Riemann–Liouville Derivative. We refer to Podlubny [9] for more details.

We have

$$\frac{d^n}{dt^n}(D^{p}_{a,-}(f)(t)) = D^{n+p}_{a,-}(f)(t). \tag{1}$$

On the contrary, we have

$$D^{p}_{a,-}\left(\frac{d^n f(t)}{dt^n}\right)(t) = D^{p+n}_{a,-}(f)(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^{j-p-n}}{\Gamma(1+j-p-n)}. \tag{2}$$

The Riemann–Liouville derivative commutes with the usual derivative if and only if $f^{(k)}(a) = 0$ for $k = 0, \dots, n-1$.

We have also the following composition formula: let $m - 1 \leq p < m$ and $n - 1 \leq q < n$; then

$$D^{p}_{a,-}(D^{q}_{a,-}(f)(t))(t) = D^{p+q}_{a,-}(f)(t) - \sum_{j=1}^{n} \left[D^{q-j}_{a,-}(f)(t)\right]_{t=a} \frac{(t-a)^{-p-j}}{\Gamma(1-p-j)},$$

$$D^{q}_{a,-}(D^{p}_{a,-}(f)(t))(t) = D^{p+q}_{a,t}(f)(t) - \sum_{j=1}^{m} \left(\left[D^{p-j}_{a,t} f(t)\right]_{t=a}\right)(t) \frac{(t-a)^{-q-j}}{\Gamma(1-q-j)}.$$

In general, we have no commutation between Riemann–Liouville derivatives. Commutation holds if and only if $f^{(j)}(a) = 0, j = 0, \dots, r-1$ with $r = \max(n, m)$ and similarly for the right fractional derivative.

1.2. *Local Fractional Derivative*

In order to avoid these difficulties, Kolvankar and Gangal [5] have introduced a notion of (right) local fractional derivative. In this section, we develop this idea. In particular, we obtain a simple analytic expression for the local fractional derivative (Theorem 1.1).

DEFINITION 1.2. Let f be a continuous function on $[a,b]$; we call a right (resp. left) local fractional derivative of f at $y \in [a,b]$ the following quantity

$$d_\sigma^\alpha f(y) = \lim_{x \to y^\sigma} D_{y,-\sigma}^\alpha [\sigma(f - f(y))](x), \tag{3}$$

for $\sigma = \pm$, respectively.

We have the following obvious properties:

(i) (Gluing) If f is differentiable at x, we have

$$\lim_{\alpha \to 1} d_\sigma^\alpha f(x) = f'(x),\ \sigma = \pm.$$

(ii) We have $d_\pm^\alpha(C) = 0$ for all $C \in \mathbb{R}$ and $\sigma = \pm$.

THEOREM 1.1. *The (right or left) local fractional derivative of* f, $d_\sigma^\alpha f(x)$, *is equal to*

$$d_\sigma^\alpha f(x) = \Gamma(1+\alpha) \lim_{y \to x^\sigma} \frac{\sigma(f(y) - f(x))}{|y-x|^\alpha}. \tag{4}$$

We then obtain the notion of α-velocity introduced by Cherbit [4] in his study of non-differentiable curves. The proof is based on a generalized Taylor's expansion theorem. We denote

$$F_\sigma(y, \sigma(x-y), \alpha) = D_{y,-\sigma}^\alpha [\sigma(f - f(y))](x). \tag{5}$$

THEOREM 1.2. *Let* f *be a continuous function such that* $d_\sigma^\alpha f(y)$ *exists for* $\alpha > 0$, $\sigma = \pm$; *then*

$$f(x) = f(y) + \sigma \frac{d_\sigma^\alpha f(y)}{\Gamma(1+\alpha)} [\sigma(x-y)]^\alpha + R_\sigma(x,y), \tag{6}$$

with

$$R_\sigma(x,y) = \sigma \frac{1}{\Gamma(1+\alpha)} \int_0^{x-y} \frac{dF_\sigma(y, \sigma t, \alpha)}{dt} (\sigma(x-y-t))^\alpha dt, \tag{7}$$

and

$$\lim_{x \to y^\sigma} \frac{R_\sigma(x,y)}{(\sigma(x-y))^\alpha} = 0.$$

Proof. We detail the proof for $\sigma = +$. The proof for $\sigma = -$ is the same. We omit the $+$ index in the following formula. We have

$$f(x) - f(y) = \frac{1}{\Gamma(\alpha)} \int_0^{x-y} \frac{F(y,t,\alpha)}{(x-y-t)^{1-\alpha}}\, dt$$
$$= \frac{1}{\Gamma(\alpha)} \left[F(y,t,\alpha) \int (x-y-t)^{\alpha-1}\, dt \right]_0^{x-y}$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^{x-y} \frac{dF(y,t,\alpha)}{dt} \frac{(x-y-t)^{\alpha}}{\alpha}\, dt.$$

Then,

$$f(x) - f(y) = \frac{d^\alpha f(y)}{\Gamma(\alpha+1)} (x-y)^\alpha$$
$$+ \frac{1}{\Gamma(\alpha+1)} \int_0^{x-y} \frac{dF(y,t,\alpha)}{dt} (x-y-t)^\alpha\, dt.$$

Hence, we have $f(x) = f(y) + \frac{d^\alpha f(y)}{\Gamma(\alpha+1)}(x-y)^\alpha + R(x,y)$. Moreover, we have

$$\frac{R(x,y)}{(x-y)^\alpha} = \frac{1}{\Gamma(\alpha+1)} \int_0^{x-y} \frac{dF(y,t,\alpha)}{dt} \left(\frac{x-y-t}{x-y} \right)^\alpha dt.$$

As $\left| \frac{x-y-t}{x-y} \right| < 1$, we obtain $\left| \frac{R(x,y)}{(x-y)^\alpha} \right| < \frac{1}{\Gamma(\alpha+1)} (|F(y, x-y, \alpha)| - |d^\alpha f(y)|)$. As $\lim_{x \to y} F(y, x-y, \alpha) = d^\alpha f(y)$, we deduce that $\lim_{x\to y} \left| \frac{R(x,y)}{(x-y)^\alpha} \right| = 0$, which concludes the proof. ∎

Theorem 1.1 follows easily from Theorem 1.2.

We then have the following notion of a local α-derivative:

DEFINITION 1.3. Let I be an open interval of $\mathbb{R}$, $\alpha \in]0,1]$ and let f be a function on I. We say that f is right (resp. left) locally α-derivative at $t_0 \in I$ if and only if the function $t \longmapsto \frac{f(t)-f(t_0)}{\sigma(\sigma(t-t_0))^\alpha}$, $\sigma = +$ (resp. $\sigma = -$), admits a limit in $\mathbb{R}$ when $t \to t_0^\sigma$.

As in the classical case, we have:

PROPOSITION 1.1. *Let f be a function on $I \subset \mathbb{R}$, $\alpha \in]0,1]$.*

(i) *If f is α-differentiable at t_0, then f is continuous at this point.*

(ii) *If $d_-^\alpha f(t_0)$ (resp. $d_+^\alpha f(t_0)$) exists, then f is left (resp. right) continuous at t_0.*

Of course, the α-right or α-left local derivatives may not exist. However, the quantities always defined are

$$
\begin{aligned}
\overline{\lim_{x\to x_0^+}} \frac{f(x)-f(x_0)}{(x-x_0)^\alpha} &= \Lambda_+^\alpha(x_0),\\
\underline{\lim_{x\to x_0^+}} \frac{f(x)-f(x_0)}{(x-x_0)^\alpha} &= \lambda_+^\alpha(x_0),\\
\overline{\lim_{x\to x_0^-}} \frac{f(x)-f(x_0)}{-(-(x-x_0))^\alpha} &= \Lambda_-^\alpha(x_0),\\
\underline{\lim_{x\to x_0^-}} \frac{f(x)-f(x_0)}{-(-(x-x_0))^\alpha} &= \lambda_-^\alpha(x_0).
\end{aligned}
\tag{8}
$$

If $\Lambda_+^\alpha(x_0)$ and $\lambda_+^\alpha(x_0)$ are finite and equal, then it is equal to the α-right local derivative at x_0. Similarly, if $\Lambda_-^\alpha(x_0) = \lambda_-^\alpha(x_0)$, then it is equal to the α-left local derivative at x_0.

For $f(x) = (x-x_0)^\alpha$ if $x > x_0$, 0 otherwise, we have $\Lambda_+^\alpha(x_0) = \lambda_+^\alpha(x_0) = 1$ and $\Lambda_-^1(x_0) = \lambda_-^1(x_0) = 0$.

EXAMPLES. Let $[a, b] = [0, 1]$, let $x_1 = 1/2$, let $x_k = 1/2 + 1/k$, and let

$$f(x) = |x - x_1| + \sum_{k=2}^{\infty} \frac{|x-x_k|^\alpha}{2^k}.$$

The function $f(x)$ is Hölderian with an exponent $\mu(x) = \alpha$ for $x \neq 1/2$ and $\mu(1/2) = 1$. We have also $d_\sigma^\alpha f(x_k) = \frac{\sigma}{2^k}$ for $k \geq 2$ and $d_\sigma^1 f(1/2) = 1 - \sum_{k=2}^{\infty} 2^{-k} \frac{\sqrt{k}}{2}$ and $d_\sigma^\nu f(x) = 0$ for $0 < \nu < \alpha$ and $x \neq x_k$, $k \geq 1$.

Let $Q = \{x_k\}_{k=1}^{\infty}$ be an ordered sequence of rational numbers in $[0, 1]$. We denote $Q = \bigcup_l Q_l$, where $\{x_i^l\} \in Q_l$ if $\frac{1}{l+1} < |x_i^l - 1/2| < l^{-1}$. We define

$$f(x) = |x - x_0| + \sum_{l=2}^{\infty} 2^{-l} \sum_{i=1}^{\infty} |x_i^l - x|^{\delta_i^l} 2^{-i},$$

$$x_0 = \frac{1}{2},\ \delta_i^l \in]0, 1[,\ \inf_{l,i} \delta_i^l = \delta_0.$$

Then $f(x)$ is Hölderian with an exponent $\mu(x) = \delta_0$ for $x \neq 1/2$ and $\mu(1/2) = 1$.

1.2.1. *Local Fractional Derivative of Order* $\alpha + n$ $(0 < \alpha < 1, n \in \mathbb{N})$

DEFINITION 1.4. Let $f \in C^n$; the local fractional derivative of order $\alpha + n$ of f is defined by

$$d_\sigma^{\alpha+n} f(x) = d_\sigma^\alpha f^{(n)}(x), \qquad \sigma = \pm. \tag{9}$$

We can give a slightly different definition by writing

$$d^{\alpha+n} f(x) = \lim_{y\to x} D^{\alpha}_{x,\sigma}\left[\sigma\left(f(x) - \sum_{k=0} \frac{f^{(k)}(y)}{k!}(x-y)^k\right)\right](x). \tag{10}$$

If f is of class C^n, then these two definitions coincide.

1.2.2. *Taylor Expansion*

We denote by $C^{<k+1}$ the set of functions of class C^k, such that the $k+1$th derivative does not exist. We have the following generalization of Thereom 1.2:

THEOREM 1.3. *Let $0 < \alpha < 1$, $f \in C^{<k+1}$ such that $d^{\alpha+k} f$ exists at y; then*

$$\begin{aligned} f(x) = f(y) + \sum_{i=1}^{k} \frac{f^{(i)}(y)}{\Gamma(i+1)}(x-y)^i \\ + \sigma \frac{d^{\alpha}_{\sigma} f^{(k+1)}(y)}{\Gamma(k+\alpha+1)}[\sigma(x-y)]^{k+\alpha} + R_{\sigma}(x, y), \end{aligned} \tag{11}$$

with

$$\lim_{x\to y^{\alpha}} \frac{R_{\sigma}(x, y)}{(\sigma(x-y))^{k+\alpha}} = 0.$$

Proof. We detail the proof for $\sigma = +$. The $\sigma = -$ case is similar. As $f \in C^{<k+1}$, the classical Taylor's expansion theorem up to order $k-1$ gives

$$f(x) = f(y) + \sum_{i=1}^{k-1} \frac{f^{(i)}(y)}{\Gamma(i+1)}(x-y)^i + \int_y^x \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t)\, dt.$$

As $f^{(k)}$ admits a derivative of order α by assumption, we have for all $t \in [y, x]$

$$f^{(k)}(t) = f^{(k)}(y) + \frac{(t-y)^{\alpha}}{\Gamma(\alpha+1)} d^{k+\alpha} f(y) + o((t-y)^{\alpha}).$$

By integrating by part $k-1$ times, we obtain

$$f(x) = f(y) + \sum_{i=1}^{k} \frac{f^{(i)}(y)}{\Gamma(i+1)}(x-y)^i + \frac{d^{\alpha} f^{(k+1)}(y)}{\Gamma(k+\alpha+1)}(x-y)^{k+\alpha} + o((x-y)^{k+\alpha}),$$

from which the result follows. ∎

1.2.3. *Another Kind of Expansion*

The Riemann–Liouville derivative of order $n+\alpha$, $n \in \mathbb{N}$, and $0 < \alpha < 1$ is defined by

$$D^{\alpha+n}_{a,-}(f)(x) = \frac{d^{n+1} I^{n+1-\alpha}_{a,-}(f)(x)}{dx^{n+1}} \quad \text{and} \quad D^{\alpha+n}_{b,+}(f)(x) = \frac{d^{n+1} I^{n+1-\alpha}_{b,+}(f)(x)}{dx^{n+1}}. \tag{12}$$

We denote

$$\mathscr{D}^{a+n}_{\sigma}(f)(y) = \lim_{x \to y^{\sigma}} D^{\alpha+n}_{y,-\sigma}[\sigma(f - f(y))](x). \tag{13}$$

A direct use of the additivity properties of Riemann–Liouville differentiation (formula (1) or (2)) within formula (13) will not give (9).

THEOREM 1.4. *Let* $\alpha \in]0,1[$, *and let* f *be a function* $(\alpha+k)$*-differentiable on* $]a,b[$ *for all* $0 \le k \le n$. *Let* $x_0 \in]a,b[$. *There exists* $\delta > 0$ *such that for all* $x \in]x_0 - \delta, x_0 + \delta[$ *we have*

$$f(x) = f(x_0) + \sum_{k=0}^{n} \sigma \frac{[\sigma(x-x_0)]^{\alpha+k}}{\Gamma(\alpha+k+1)} \mathscr{D}^{\alpha+k}_{\sigma} f(x_0) + R_{n+1,\sigma}(x,y) \tag{14}$$

with

$$R_{n+1,\sigma}(x,y) = \frac{\sigma}{\Gamma(\alpha+n+1)} \int_0^{x-x_0} \frac{d^{n+1} F_{\sigma}(x_0, \sigma t, \alpha)}{dt^{n+1}} (\sigma(x-x_0-t))^{\alpha+n}\, dt,$$

and

$$\lim_{x \to x_0} \left| \frac{R_{n+1,\sigma}(x,x_0)}{(\sigma(x-x_0))^{\alpha+n}} \right| = 0.$$

Proof. We perform an integration by part from (6). We obtain

$$f(x) - f(y) = \frac{d^{\alpha} f(y)}{\Gamma(\alpha+1)} (x-y)^{\alpha} + \frac{(x-y)^{\alpha+1}}{\Gamma(\alpha+2)} \frac{dF(y,t,\alpha)}{dt}\bigg|_{t=0} + \frac{1}{\Gamma(\alpha+2)} \int_0^{x-y} \frac{d^2 F(y,t,\alpha)}{dt^2} (x-y-t)^{\alpha+1}\, dt.$$

$F(y, x-y, \alpha) = D^{\alpha}_{y,-}[(f - f(y))](x)$, where y is a given constant. We denote $x - y = t$. We then have $F(y,t,\alpha) = D^{\alpha}_{y,-}[(f-f(y))](t+y)$; hence

$$\frac{dF(y,t,\alpha)}{dt} = \frac{d}{dt}(D^{\alpha}_{y,-}[(f-f(y))](t+y)).$$

As $x - y = t$ and $dx = dt$, we have $\frac{dF(y,t,\alpha)}{dt}|_{t=0} = \frac{d}{dx}(D^{\alpha}_{y,-}[(f - f(y))](x))|_{x=y}$. Properties of the Riemann–Liouville derivative [8, 9] allow us to write

$$\frac{d}{dx}(D^{\alpha}_{y,-}[(f - f(y))](x))|_{x=y} = D^{\alpha+1}_{y,-}[(f - f(y))](x)|_{x=y}.$$

By definition of $\mathscr{D}^{\alpha+n}$, we have $\frac{dF(y,t,\alpha)}{dt}|_{t=0} = \mathscr{D}^{\alpha+1}f(y)$, which gives

$$f(x) - f(y) = \frac{d^{\alpha}f(y)}{\Gamma(\alpha+1)}(x-y)^{\alpha} + \frac{1}{\Gamma(\alpha+2)}(x-y)^{\alpha+1}\mathscr{D}^{\alpha+1}f(y) + \frac{1}{\Gamma(\alpha+2)}\int_0^{x-y}\frac{d^2F(y,t,\alpha)}{dt^2}(x-y-t)^{\alpha+1}\,dt.$$

By integrating by part up to order n, we obtain

$$f(x) - f(y) = \sum_{k=0}^{n}\frac{(x-y)^{\alpha+k}}{\Gamma(\alpha+k+1)}\mathscr{D}^{\alpha+k}f(y) + \frac{1}{\Gamma(\alpha+n+1)}\int_0^{x-y}\frac{d^{n+1}F(y,t,\alpha)}{dt^{n+1}}(x-y-t)^{\alpha+n}\,dt.$$

Denoting by $R_{n+1}(x,y)$ the remainder, we see that

$$\left|\frac{R_{n+1}(x,y)}{(x-y)^{\alpha+n}}\right| < \frac{1}{\Gamma(\alpha+n+1)}\left(\left|\frac{d^n}{dt^n}F(y,t;\alpha)\right|_{t=x-y} - \left|\mathscr{D}^{\alpha+n}f(y)\right|\right).$$

We deduce that $\lim_{x\to y}\left|\frac{R_{n+1}(x,y)}{(x-y)^{\alpha+n}}\right| = 0$. ∎

Remark. There exists a Taylor's expansion theorem for the Riemann-Liouville derivative (see [11]). This result cannot be used to study the local behaviour of non-differentiable functions. Indeed, it depends on an implicit function. Moreover, the remainder is not controlled.

THEOREM 1.5. *Let $\alpha \in]0,1[$, let $\sigma = \pm$, and let I be an interval of $\mathbb{R}$.*

(i) *If f is differentiable on I, then for all $x \in I$ we have $d^{\alpha}_{\sigma}f(x) = 0$.*

(ii) *The reverse of* (i) *is wrong.*

Proof. (i) f is differentiable on I, then we have

$$d^{\alpha}_{\sigma}f(x) = \Gamma(1+\alpha)\lim_{y\to x^{\sigma}}\frac{\sigma(f(y)-f(x))}{|y-x|^{\alpha}}.$$

$$= \Gamma(1+\alpha)\lim_{y\to x^{\sigma}}\frac{\sigma(f(y)-f(x))}{|y-x|}\lim_{y\to x^{\sigma}}|y-x|^{1-\alpha}.$$

$$= \Gamma(1+\alpha)f'(x)\lim_{y\to x^{\sigma}}|y-x|^{1-\alpha} = 0.$$

(ii) As $d^{\alpha}_{\sigma}(I^{\alpha}_{a,-\sigma}f(x))(x) = 0$ for all f, we deduce (ii). ∎

1.3. *Non-differentiability and α-Derivative*

Let $f(t)$ be a continuous function on $[a, b]$. We remark that $d_+^\alpha f(x) \neq d_-^\alpha f(x)$ in general. In the differentiable case, we have (by (ii), Sect. 1.2), $d_+^1 f(x) = d_-^1 f(x)$. In other words, the non-differentiability of a function is characterized by the existence of right and left local fractional derivatives, which carry different information on the local behaviour of the function. It is then necessary to introduce a new notion which takes into account these two data.

DEFINITION 1.5. Let $f(t)$ be a continuous function on $[a, b]$ such that $d_\sigma^\alpha f(y)$ exists for $\sigma = \pm$ and $y \in [a, b]$. We define the α-derivative of f at y, and we denote by $f^{(\alpha)}(y)$ the quantity

$$f^{(\alpha)}(y) = \frac{d_+^\alpha f(y) + d_-^\alpha f(y)}{2} + i\frac{d_+^\alpha f(y) - d_-^\alpha f(y)}{2}, \quad \text{where } i^2 = -1. \tag{15}$$

When $\alpha = 1$ and f is non-differentiable, but possesses a right and left derivative, we find the notion of differential time symmetry breaking discussed by Nottale [7] as a consequence of the“fractal” nature of space-time.

When f is differentiable, we have $f^{(1)}(y) = f'(y)$. If f is 1-differentiable, the non-differentiable is equivalent to the existence of an imaginary part for the 1-derivative.

DEFINITION 1.6. A function f is said α-differentiable if the α-derivative exists at all points.

2. α-DERIVATIVE PROPERTIES

We give several properties of the local fractional derivative defined in Section 1.2. We deduce general properties of the α-derivative introduced in Section 1.3.

2.1. *Local Fractional Derivative Properties*

In the following, the proofs are always given for $d_+^\alpha f$. The approach is exactly the same for $d_-^\alpha f$. We omit the index + and we will denote $d^\alpha f$ for $d_+^\alpha f$.

PROPOSITION 2.1. *Let f and g be two continuous functions on $[a, b]$, α-differentiable at x_0, $0 < \alpha < 1$, and λ a real number. Then $f + g$, λf, and fg are α-differentiable at x_0 and we have for $\sigma = \pm$:*

(i) $d_\sigma^\alpha (f + g)(x_0) = d_\sigma^\alpha (x_0) + d_\sigma^\alpha g(x_0)$.

(ii) $d_\sigma^\alpha \lambda f(x_0) = \lambda d_\sigma^\alpha f(x_0)$.

(iii) $d^{\alpha}_{\sigma}(fg)(x_0) = d^{\alpha}_{\sigma}f(x_0)\cdot g(x_0) + f(x_0)\cdot d^{\alpha}_{\sigma}g(x_0)$.

Proof. Points (i) and (ii) are obvious. For (iii), we have

$$d^{\alpha}(fg)(x_0) = \Gamma(\alpha)\lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{(x-x_0)^{\alpha}}g(x_0) + f(x)\frac{g(x)-g(x_0)}{(x-x_0)^{\alpha}}\right).$$

We deduce Proposition 2.1. ∎

PROPOSITION 2.2. *Let g and f be continuous functions on $[a,b]$ and $[g(a),g(b)]$, respectively. Let $x_0 \in]a,b[$ such that $d^{\beta}_{\sigma}g(x_0)$ and $d^{\alpha}_{\sigma s^{\sigma}}f(g(x_0))$ exist, with $s^{\sigma} = \mathrm{sign}(d^{\beta}_{\sigma}g(x_0))$, $\alpha,\beta \in]0,1]$, $\sigma = \pm$. Then, we have*

$$d^{\alpha\beta}_{\sigma} f\circ g(x_0) = s^{\sigma} d^{\alpha}_{\sigma s^{\sigma}} f(g(x_0))|d^{\beta}_{\sigma}g(x_0)|^{\alpha}. \tag{16}$$

Proof. We have

$$\frac{\sigma(f(g(x))-f(g(x_0)))}{(\sigma(x-x_0))^{\alpha\beta}} = s^{\sigma}\frac{\sigma s^{\sigma}(f(g(x))-f(g(x_0)))}{(\sigma s^{\sigma}(g(x)-g(x_0)))^{\alpha}}\left(s^{\sigma}\frac{\sigma(g(x)-g(x_0))}{(\sigma(x-x_0))^{\beta}}\right)^{\alpha}.$$

Moreover, when $x \to x_0^{\sigma}$ we have $g(x) \to (g(x_0))^{\sigma s^{\sigma}}$ because

$$g(x) = g(x_0) + \sigma s^{\sigma}\frac{|d^{\beta}_{\sigma}g(x_0)|}{\Gamma(1+\beta)}(\sigma(x-x_0))^{\beta}(1+R_{\beta}(x)),$$

where $R_{\beta}(x) \to 0$ when $x \to x_0^{\sigma}$. We deduce that $d^{\alpha\beta}_{\sigma} f\circ g(x_0) = s^{\sigma} d^{\alpha}_{\sigma s^{\sigma}} f(g(x_0))(s^{\sigma} d^{\beta}_{\sigma} g(x_0))^{\alpha}$. This concludes the proof. ∎

PROPOSITION 2.3. *Let f and g be continuous functions on $[a,b]$ such that g is not zero on $[a,b]$. If f and g admit a local fractional derivative of order $0<\alpha<1$ on $[a,b]$ at x_0, then*

$$d^{\alpha}_{\sigma}\left(\frac{f}{g}\right)(x_0) = \frac{d^{\alpha}_{\sigma}f(x_0)\cdot g(x_0) - f(x_0)\cdot d^{\alpha}_{\sigma}g(x_0)}{g^2(x_0)}. \tag{17}$$

Proof. We have

$$d^{\alpha}\left(\frac{f}{g}\right)(x_0) = \Gamma(\alpha)\lim_{x\to x_0}\frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0)}{(x-x_0)^{\alpha}};$$

hence

$$d^{\alpha}_{\sigma}\left(\frac{f}{g}\right)(x_0) = \Gamma(\alpha)\lim_{x\to x_0}\frac{\left(\frac{f(x)-f(x_0)}{(x-x_0)^{\alpha}}g(x_0) - f(x)\frac{g(x)-g(x_0)}{(x-x_0)^{\alpha}}\right)}{g(x)g(x_0)}.$$

We deduce the proposition. ∎

2.2. α-Derivative Properties

In the following, we give several properties of α-derivatives (Definition 1.4). Proofs follow easily from the corresponding results in Section 2.1.

PROPOSITION 2.4. *Let f and g be continuous functions on $[a, b]$, such that $d_\sigma^\alpha f(y)$ and $d_\sigma^\alpha g(y)$ exist for $\sigma = \pm$, $y \in [a, b]$, $0 < \alpha < 1$, and λ a real number. Then $f + g$, λf, and fg are α-differentiable at y and we have*

(i) $(f + g)^\alpha(y) = f^\alpha(y) + g^\alpha(y)$.

(ii) $(\lambda f)^{(\alpha)}(y) = \lambda f^{(\alpha)}(y)$.

(iii) $(fg)^{(\alpha)}(y) = f^{(\alpha)}(y) \cdot g(y) + f(y) \cdot g^{(\alpha)}(y)$.

PROPOSITION 2.5. *Let g and f be continuous functions on $[a, b]$ and $[g(a), g(b)]$, respectively. Let $x_0 \in]a, b[$ such that $d_\sigma^\beta g(x_0)$ and $d_{s^\sigma}^\alpha f(g(x_0))$ exist, $\alpha, \beta \in]0, 1]$. Then, we have*

$$\begin{aligned}(f \circ g)^{(\alpha\beta)}(x_0) = {} & f^{(\alpha)}(g(x_0)) \cdot s^+|d_+^\beta g(x_0)|^\alpha \\ & - d_{s-}^\alpha f(g(x_0)) \cdot \left[\frac{s^+|d_+^\beta g(x_0)|^\alpha - s^-|d_-^\beta g(x_0)|^\alpha}{2}\right. \\ & \left. - \frac{i}{2}(s^+|d_+^\beta g(x_0)|^\alpha - s^-|d_-^\beta g(x_0)|^\alpha)\right], \end{aligned} \tag{18}$$

where $s^\sigma = \text{sign}(d_\sigma^\beta g(x_0))$, $\sigma = \pm$.

Remarks. (i) If g is differentiable at x_0, with $s = \text{sign}(g'(x))$, then

$$(f \circ g)^{(\alpha)}(x_0) = s \cdot f^{(\alpha)}(g(x_0)) \cdot (s \cdot g'(x_0))^\alpha. \tag{19}$$

(ii) If f is differentiable at $g(x_0)$ and g is α-differentiable at x_0, then

$$d_\sigma^\alpha(f \circ g)(x_0) = f'(g(x_0)) \cdot d_\sigma^\alpha g(x_0). \tag{20}$$

PROPOSITION 2.6. *Let f and g be continuous functions on $[a, b]$, such that $d_\sigma^\alpha f(y)$ and $d_\sigma^\alpha g(y)$ exist for $\sigma = \pm$, $y \in [a, b]$, $0 < \alpha < 1$. If g is not zero on $[a, b]$ then*

$$\left(\frac{f}{g}\right)^{(\alpha)}(y) = \frac{f^{(\alpha)}(y) \cdot g(y) - f(y) \cdot g^{(\alpha)}(y)}{g^2(y)}. \tag{21}$$

2.3. About Functions of Class $\mathscr{C}^\alpha$

We define the set of functions of class $\mathscr{C}^\alpha$, $\alpha \in]0, 1]$, and its precise properties.

2.3.1. *On the Set* $\mathscr{C}^\alpha$

Let Ω be an open interval of $\mathbb{R}$, $\alpha \in]0,1]$; we denote by $\mathscr{C}^\alpha(\overline{\Omega})$ the set of continuous functions on $\overline{\Omega}$ which are α-differentiable on Ω.

Remark. We have $C^1 \subset \mathscr{C}^1$. Indeed, when $\alpha = 1$ and $d_+^1 f(x) = d_-^1 f(x)$ for all x, then f is differentiable.

We can define a norm on $\mathscr{C}^\alpha$ by

$$\| f \|_\alpha = \| f(x) \| + \sup\{\| d_+^\alpha f(x) \|, \| d_-^\alpha f(x) \|, x \in \mathbb{R}\}. \tag{22}$$

LEMMA 2.1. *Let* $\gamma, \alpha \in]0,1]$. *For all* $\gamma < \alpha$, *we have* $\mathscr{C}^\alpha \subset \mathscr{C}^\gamma$.

Proof. Let $f \in \mathscr{C}^\alpha$. We have

$$f(x) = f(x_0) + \sigma(\sigma(x - x_0))^\alpha \left(\frac{d_\sigma^\alpha f(x_0)}{\Gamma(1+\alpha)} + R_\sigma^\alpha(x) \right),$$

where $R_\sigma^\alpha(x) \to 0$ when $x \to x_0^\sigma$. We deduce

$$\frac{f(x) - f(x_0)}{(\sigma(x - x_0))^\gamma} = \sigma(\sigma(x - x_0))^{\alpha-\gamma} \left(\frac{d_\sigma^\alpha f(x_0)}{\Gamma(1+\alpha)} + R_\sigma^\alpha(x) \right).$$

If $\gamma < \alpha$, then $d_\sigma^\gamma f(x_0) = 0$. This concludes the proof. ∎

Properties of α-differentiation give:

THEOREM 2.1. *If* $f \in \mathscr{C}^\alpha$ *and* $g \in \mathscr{C}^\beta$ *with* $\beta > \alpha$, *we have*

(i) $fg \in \mathscr{C}^\alpha$,

(ii) $f + g \in \mathscr{C}^\alpha$,

(iii) $f/g \in \mathscr{C}^\alpha$ *if* $g \neq 0$.

Composition of functions has a complicated behaviour with respect to differentiation. Precisely, we

THEOREM 2.2. *If* $f \in \mathscr{C}^\alpha$ *and* $g \in \mathscr{C}^\beta$, *then* $f \circ g \in \mathscr{C}^{\alpha\beta}$.

Proof. This follows easily from Proposition 2.2. ∎

In other words, the set $\mathscr{C}^\alpha$ is only stable for composition by elements of $\mathscr{C}^1$.

2.3.2. *Critical Order and Hölder Exponent*

Let Ω be an open interval of $\mathbb{R}$, $\gamma \in]0, 1]$. We denote by

$$C^{0,\gamma}(\overline{\Omega}) = \left\{ f \in C(\overline{\Omega}), \qquad \sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^{\gamma}} < \infty \right\}$$

the set of Hölderian functions with Hölder exponent γ. For more details on these sets, we refer to [12–14].

LEMMA 2.2. *We have* $C^{0,\gamma}(\overline{\Omega}) \subset \mathscr{C}^{\alpha}(\overline{\Omega})$ *for* $\gamma > \alpha$.

Proof. Indeed, for all $f \in C^{0,\gamma}(\overline{\Omega})$, $\gamma > \alpha$, we have $d^{\alpha}_{\sigma} f = 0$. ∎

As $C^{0,\gamma}(\overline{\Omega}) \subset C^{0,\alpha}(\overline{\Omega})$ if $\gamma > \alpha$, we are led to introduce the notion of *critical exponent*:

DEFINITION 2.1. Let Ω be an open interval of $\mathbb{R}$. We call the critical exponent at $x_0 \in \Omega$ the largest α such that f is Hölderian with a Hölder exponent α at x_0.

A direct consequence of the Taylor's expansion theorem is that $f \in C^{0,\gamma}(\overline{\Omega})$ if f is γ-differentiable with $d^{\gamma}_{\sigma} f \neq 0$. We then introduce the notion of critical order:

DEFINITION 2.2. Let Ω be an open interval of $\mathbb{R}$. We call the critical order of f at $x_0 \in \Omega$ the smallest $\alpha \in]0, 1]$ such that f is α-differentiable at x_0 and $d^{\alpha}_{\sigma} f(x_0) \neq 0$.

We then have the following theorem:

THEOREM 2.3. *Let f be a function of critical exponent α at x_0; then the critical order of f at x_0 is (if it exists) α.*

Proof. This is a simple consequence of the definitions. ∎

We have also:

THEOREM 2.4. *If f has a critical order α at x_0, then f is Hölderian with a critical exponent α.*

Proof. As f has for critical order α at x_0, we have $d^{\alpha}_{\sigma} f(x_0) \neq 0$ and $f \in C^{0,\alpha}(\overline{\Omega})$. Assume that f has for critical exponent $\gamma > \alpha$. Then, we have $d^{\alpha}_{\sigma} f(x_0) = 0$. We have a contradiction. Hence, the critical exponent is α. ∎

3. LOCAL GEOMETRY OF $\mathscr{C}^\alpha$ FUNCTIONS

3.1. *Local Study*

Non-differentiable curves are characterized by the appearance of infinitely many new structures when one applies successive zooms. The study of local geometrical properties of these kind of curves is then difficult. In this section, we give some results in this direction.

3.1.1. *About Infinitesimals*

Let Δx be an arbitrary small increment; we have $\forall \alpha \in]0,1[$, $\frac{\Delta x}{(\sigma \Delta x)^\alpha} \to 0$ when $\Delta x \to 0$, with $\sigma = \pm 1$ depending on the sign of Δx.

For all $0 < \alpha_1 < \alpha_2 < \alpha_3 < 1$, we have

$$0 < \Delta x < (\Delta x)^{\alpha_3} < (\Delta x)^{\alpha_2} < (\Delta x)^{\alpha_1} < 1,$$

which means that Δx depends also on α.

The local fractional derivative can be expressed via $(\Delta x)^\alpha$ by

$$d_\sigma^\alpha f(x_0) = \Gamma(1+\alpha) \lim_{(\sigma\Delta x)^\alpha \to 0} \frac{\sigma(f(x_0+\sigma\Delta x) - f(x_0))}{(\sigma\Delta x)^\alpha}. \tag{23}$$

3.1.2. *Irregularity Criterion*

Lebesgue's theorem allows us to characterize non-differentiable functions.

THEOREM 3.1. *Let Ω be an open interval of $\mathbb{R}$, and let f be a continuous function.*

(i) *If for all $x, y \in \Omega$, f is monotone, then f is differentiable almost everywhere.*

(ii) *f is non-differentiable on Ω if and only if the set*

$$\{O \text{ an open interval of } \Omega / \forall\, x, y \in O,$$
$$\operatorname{sign}(\sigma(f(x) - f(y))) = cst,\ \sigma = \operatorname{sign}(x-y),\}$$

is empty, where $\sigma = \pm 1$.

Proof. (i) This is the content of Lebesgue's theorem. For (ii), we assume that f is non-differentiable on Ω and that there exists an open interval O of Ω such that $\operatorname{sign}(\sigma(f(x)-f(y))) = cst$. Then f is monotone on O; hence f is differentiable almost everywhere on $O \subset \Omega$, which is a contradiction. If f is differentiable on Ω, there exists an open interval $O \subset \Omega$ such that for all $x, y \in O$, $\operatorname{sign}(\sigma(f(x)-f(y))) = cst$. We deduce the theorem. ∎

For all $\delta > 0$, we denote by $J_x^-(\delta)$ (resp. $J_x^+(\delta)$) the interval $]x-\delta, x]$ (resp. $[x, x+\delta[$).

PROPOSITION 3.1. *Let f be a continuous function on $[a,b](a<b)$, α-differentiable at $x_0 \in]a,b[(\alpha \in]0,1[)$.*

(i) $\sigma d_-^\alpha f(x_0) < 0 \Longleftrightarrow \exists \delta > 0$ *such that* $\forall x \in J_{x_0}^-(\delta)$; *we have* $\sigma(f(x) - f(x_0)) > 0$, $\sigma = \pm$.

(ii) $\sigma d_+^\alpha f(x_0) < 0 \Longleftrightarrow \exists\, \delta > 0$ *such that* $\forall x \in J_{x_0}^+(\delta)$; *we have* $\sigma(f(x) - f(x_0)) < 0$, $\sigma = \pm$.

Proof. By Theorem 1.2, we have, for $\delta > 0$ sufficiently small and $x \in J_{x_0}^\sigma(\delta)$,

$$f(x) - f(x_0) = \sigma s^\sigma \mid d_\sigma^\alpha f(x_0) \mid (\sigma(x - x_0))^\alpha (1 + r_\sigma(x)),$$

with $s^\sigma = \text{sign}(d_\sigma^\alpha f(x_0))$ and $r_\sigma(x) \to 0$ when $x \to x_0^\sigma$. The sign of $\sigma(f(x) - f(x_0))$ is s_σ. This concludes the proof. ∎

PROPOSITION 3.2. *Let f be a continuous function on $[a,b]$ which is α-differentiable at $x_0 \in]a,b[$, $\alpha \in]0,1[$.*

(i) *if $d_-^\alpha f(x_0) \geq 0$ and $d_+^\alpha f(x_0) \leq 0 \Longleftrightarrow x_0$ is a local maximum.*

(ii) *if $d_-^\alpha f(x_0) \leq 0$ and $d_+^\alpha f(x_0) \geq 0 \Longleftrightarrow x_0$ is a local minimum.*

Proof. We first prove (i). $\rightarrow$ follows from Proposition 3.1. For $\leftarrow$ let $x_0 \in]a,b[$ be a local maximum. There exists $\delta > 0$ such that for all $x \in]x_0 - \delta, x_0 + \delta[$, we have $f(x) - f(x_0) \leq 0$. For all $x \in]x_0 - \delta, x_0]$, the ratio $\frac{f(x)-f(x_0)}{-(x_0-x)^\alpha} \geq 0$. By taking the limit, we obtain $d_-^\alpha f(x_0) \geq 0$. For all $x \in [x_0, x_0 + \delta[$, as $f(x) - f(x_0) \leq 0$, we deduce that $\frac{f(x)-f(x_0)}{(x-x_0)^\alpha} \leq 0$, and we obtain $d_+^\alpha f(x_0) \leq 0$. The proof of (ii) is similar. ∎

GENERALIZED ROLLE'S THEOREM. *Let f be a continuous and α-differentiable function on $]a,b[$, $\alpha \in]0,1[$, such that $f(a) = f(b)$; then there exists a point $c \in]a,b[$ such that*

$$d_-^\alpha f(c) \geq 0 \text{ and } d_+^\alpha f(c) \leq 0 \quad \text{or} \quad d_-^\alpha f(c) \leq 0 \text{ and } d_+^\alpha f(c) \geq 0. \tag{24}$$

Proof. The proof follows from Proposition 3.2 as in the classical case. ∎

THEOREM 3.2. *Let f and g be continuous and α-differentiable functions on $]a,b[$, $\alpha \in]0,1[$; then there exists a point $c \in]a,b[$ such that*

$$\sigma[f(b) - f(a)] d_-^\alpha g(c) \geq \sigma[g(b) - g(a)] d_-^\alpha f(c),$$

$$\sigma[f(b) - f(a)] d_+^\alpha g(c) \leq \sigma[g(b) - g(a)] d_+^\alpha f(c),$$

where $\sigma = \pm$.

Proof. We consider $h(x) = f(x)(g(b) - g(a)) - (f(b) - f(a))g(x)$. The function h is continuous and α-differentiable on $]a,b[$. As $h(a) = h(b)$, we conclude the proof using Rolle's theorem. ∎

4. CONCLUSION

The local fractional derivative allows us to obtain precise results on the behaviour of non-differentiable functions. We refer to [2] for an application of this formalism to scale divergence of graph of functions.

The α-derivative can be used to study irregular objects. For example, one can define a generalized tangent space to non-differentiable manifolds (see [3], Chap. 3).

These tools already apply to many physical problems: irregular signals [6], fractional Brownian motion [4], and scale relativity [3].

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