International Journal of Biomathematics and Biostatistics,
Vol. 1, No. 1, January-June 2010, pp. 23-31

IJBB
Serials Publications
New Delhi

Introduction to Embedding of Lagrangian Systems

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ABSTRACT: This paper is an introduction to the idea of *embedding* of ordinary or partial differential equations.

1. INTRODUCTION

This paper is an introduction to the idea of *embedding* of ordinary or partial differential equations developed in ([6], [7], [3]) in different contexts. Embedding theories take their origin in the following, not exhaustive list of problems:

- *Bath of a given biological or physical model:* In many problems of biology or physics the model if first constructed by neglecting or forgetting the particular environment of the experiment which is called a bath. For example, when one deals with the Davidov's model of energy transfer in proteins there exists a phonon bath. In general the model without the bath is a given ordinary or partial differential equation. A natural question is then to look for the new equation of the dynamics when the bath is taken into account.
- *Turbulence:* Fluid dynamics is modeled by partial differential equations. Solutions of these equations must be sufficiently smooth. However, there exits *turbulent* behavior which correspond to very irregular trajectories. If the underlying equation has a physical meaning, then one must give a sense to this equation on irregular functions. This remark is the starting point of Jean Leray's work on fluid mechanics [11]. He introduces what he calls *quasi-derivation* and the notion of weak-solutions for PDE. This first work has a long history and issue going trough the definition of Laurent Schwartz's *distribution* and *Sobolev spaces*. We refer to [1] for an overview of Jean Leray's work in this domain.
- *Deformation Quantization problems:* The problem here is to go from classical mechanics to quantum mechanics trough a deformation involving the Planck constant. Roughly speaking, we have a one parameter h family of spaces and operators such that they reduce to usual spaces and operators when h goes to zero. For example, we can look for a deformation of the classical derivative using its algebraic characterization trough the Leibniz rule. Another way is following L. Nottale [13] to assume that the space-time at the atomic scale is a non-differentiable manifold. In that case, we obtain a one parameter smooth deformation of space-time by smoothing at different scales. The main problem is then to look for the deformation of the classical derivative during this process. We refer to [2] and [3] for more details.
- *Long term behavior of the Solar-system:* The dynamics of the Solar system is usually modeled by a n-body problem. However, the study of the long-term behavior must include several perturbations terms, like tidal effects, perturbations due to the oblateness of the sun, general relativity effects etc.

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We do not know the whole set of perturbations which can be of importance for the long term dynamics. In particular, it is not clear that the remaining perturbations can be modeled using ordinary differential equations. Most of stability results uses in the Solar system dynamics make this assumption implicitly [12]. An idea is to try to look for the dynamics of the initial equation on more general objects like stochastic processes, by extending the ordinary derivative. Then one can look for the stochastic perturbation of the underlying stochastic equation which contains the original one. As a consequence, we can provide a set of dynamical behaviors which have a strong significance being stable under very general perturbation terms. This strategy is developed in [6].

These problems although completely distinct have a *common core*: we need to define what is the natural or canonical analogue of a given differential equation on a new vector space. This space can be formed of stochastic processes, non-differentiable functions, discrete data, etc. In order to obtain such analogue we must define what is the correct equivalent of the classical derivative in this new setting. The usual way to look for a PDE in the sense of Schwartz's distribution theory [14] is an example of such a procedure.

The plan of the paper is as follow: In section 2, we detail the general structure of an embedding theory and we illustrate each notion by an example in the stochastic, discrete or Schwartz's cases. In section 3 we discuss the application of embedding theories on Lagrangian systems. In section 4 we review results about the persistence of symmetries of the initial differential equation on the embedded one. In particular, we discuss the embedded version of Noether's theorem for Lagrangian systems. Section 5 we discuss with more details the stochastic embedding theory defined in [6].

2. EMBEDDING THEORIES

We denote by $C^0([a, b]; \mathbb{R}^d)$ the set of continuous functions $q : t \in [a, b] \to \mathbb{R}^d$, $d \in \mathbb{N}^*$. The general scheme underlying embedding theories of ordinary or partial differential equations is the following:

- Fix a vector space V and a mapping $\iota : C^0([a, b]; \mathbb{R}^d) \to V$.
- Extend differential operators over V.
- Extend ordinary or partial differential equations over V.

As examples of vector spaces V we have:

- *Discrete case:* $V = (\mathbb{R}^d)^N$ with $N \in \mathbb{N}$. For each functions $q \in C^0([a, b], \mathbb{R}^d)$ we associate the discrete analogue $\{q_i = q(t_i)\}$ where $t_i = a + \frac{i}{N}(b - a)$. The mapping ι is then a discretization mapping.
- *Stochastic case:* Let $T > 0$, $\nu > 0$ and $d \in \mathbb{N}^*$. Let L be the set of all measurable functions $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the following hypothesis: There exists $K > 0$ such that for all $x, y \in \mathbb{R}^d$: $\sup_t |f(t, x) - f(t, y)| \leq K|x - y|$ et $\sup_t |f(t, x)| \leq K(1 + |x|)$. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space on which a family $(W^{(b, \sigma)})_{(b, \sigma) \in L \times L}$ of Brownian motions indexed by $L \times L$ is defined. If $b, \sigma \in L$, we denote by $\mathcal{P}^{(b, \sigma)}$ the natural filtration associated to $W^{(b, \sigma)}$. We denote by $V = \Lambda_1$ the space of all diffusion processes of the form

  $$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t^{(b, \sigma)},$$

  $X_0 = X^0$ where $X^0 \in L^2(\Omega)$ and $(b, \sigma) \in L \times L$, such that for all $t \in (0, T)$, X_t admits a density $p_t(\cdot)$ and other technical properties (see [6]).

  The mapping ι is defined for all $q \in C^0([a, b]; \mathbb{R}^d)$ by $\iota(q)_t(\omega) = q(t)$ for all $\omega \in \Omega$.
- *Schwartz's distributions:* Let U be an open set of $\mathbb{R}$. $V = \mathcal{D}'(U)$ the set of distributions on U. In this case, the mapping ι is defined for all $q \in C^0([a, b]; \mathbb{R})$ by $\iota(q) : \phi \to \int_U q \cdot \phi$.

As examples of extension for the classical derivative, we have:

- *Discrete case*: Let $Q = \{q_i\}_{i=0,\dots,N}$ be an element of V. We denote by $\Delta_+ : V \to V$ the mapping defined by

$$[\Delta_+(Q)]_i = \frac{Q_{i+1} - Q_i}{h}$$

with $h = 1/N$, $i = 0, \dots, N-1$ and $[\Delta_+(Q)]_N = 0$. The mapping Δ_+ is the classical forward discretization of the classical derivative.

- *Stochastic case:* For X_t satisfying some technical properties (see [6]) the Nelson's stochastic forward derivative

$$DX_t := \lim_{h \to 0^+} E\left[\frac{X_{t+h} - X_t}{h} \mid \mathcal{P}_t\right] = b(t, X_t)$$

exists in $L^2(\Omega)$ for almost all $t \in (0, T)$.

- *Schwartz's distributions:* We define an operator $D : \mathcal{D}'(U) \to \mathcal{D}'(U)$ by $D(S)(\phi) = -S(\phi')$ for all $S \in \mathcal{D}'(U)$. We remark that we have $\iota(f') = D(\iota(f))$.

Using the vector space V and the operator D we can associated to a given differential operator $P = \sum_{i=1}^{k} a_i(\cdot) \cdot \frac{d^i}{dt^i}$ an analogue acting on a subset $V_k \subset V$ consisting of $X \in V_k$ such that $D^i X$ exists for $i = 1, \dots, k$, where $D^i = D \circ \dots \circ D$, it-th times, by

$$P_{V,D} = \sum_{i=1}^{k} a_i(\cdot) \cdot D^i. \tag{1}$$

As a consequence, an ordinary differential equation associated to a differential operator P and given by $P \cdot q = 0$ for $q \in C^k([a, b]; \mathbb{R}^d)$ can be defined over V_k by $P_{V,D} : Q = 0$ where $Q \in V_k$. Such equation is called the *embedded* version of the initial differential equation over V using D.

As an example, we consider the one dimensional *Newton's equation*

$$\ddot{q} = -\nabla V(q), \tag{2}$$

where $q \in C^2(\mathbb{R}, \mathbb{R})$ and $V : \mathbb{R} \to \mathrm{R}$ is the *potential*.

- *Discrete embedded Newton's equation:* Using the discrete embedding $((\mathbb{R}^d)^N, \Delta_+)$ we obtain

$$\frac{q_i - 2q_{i-1} + q_{i-2}}{h^2} = -\nabla V(q_i), \qquad i = 0, \dots, N, \tag{3}$$

for $Q = \{q_i\}_{i=0,\dots,N}$.

- *Stochastic Newton's equation:* Using the stochastic embedding (Λ_1, D) we obtain

$$D^2 X_t = -\nabla V(X_t), \tag{4}$$

which was studied by Thieullen and Zambrini [15] using *reciprocal processes*.

3. EMBEDDING, LAGRANGIAN SYSTEMS AND COHERENCE

A natural question arising from the previous formalism concerns the *significance* of the new equation.

The embedding of a differential equation is constructed from a differential operator associated to the equation. However, the form of this operator is not an intrinsic object. It depends mainly on the coordinates system which is used to described the dynamics. For classical ordinary differential equations the action of a change of coordinates $p = h(q)$, where h is a diffeomorphism is well understood. We denote by q_t the solutions of the equation $O \cdot q = 0$ where O is a differential operator and $O_h \cdot p = 0$ the new equation under the change of variable $p = h(q)$. The solutions of the two equations are conjugated, i.e. $h \circ q_t = p_t \circ h$. Using an embedding (V, D) we obtain two embedded equations $O_{V,D} \cdot Q = 0$ and $O_{h,V,D} \cdot P = 0$. However, the solutions of these two embedded equations are not in general conjugated or even related in a simple way.

In order to obtain embedded equations which possess a more intrinsic significance, we look for a characterization which is coordinates invariant. A natural class of such equations is given by Lagrangian systems. The solutions of a Lagrangian systems corresponds to extremal of a functional

$$\mathcal{L}(q) = \int_a^b L(q(t), \dot{q}(t), t)\, dt,$$

where $L : (x, v, t) \mapsto L(x, v, t)$ is a function, and $q : t \in [a; b] \to \mathbb{R}^d$, $d \geq 1$ is C^1. The extremal of such functional corresponds to the solutions of the second order differential equation

$$\frac{\partial L}{\partial x}(q, \dot{q}, t) - \frac{d}{dt}\left(\frac{\partial L}{\partial v}(q, \dot{q}, t)\right) = 0,$$

called the *Euler-Lagrange equation* (or (*EL*) for short).

Under embedding we can give a meaning to these functional over V that we denote by $\mathcal{L}_D(Q)$, $Q \in V$. For example, we have:

- *Discrete case:* Using the quadrature formula $\sum_{k=0}^{N-1} f(t_i)\, h$ for the discretization of the integral $\int_a^b f(t)\, dt$, we obtain

$$\mathcal{L}_h(Q) = \sum_{k=0}^{N-1} hL\left(q_k, \frac{q_{k+1} - q_k}{h}\right), \tag{5}$$

  where $Q = (q_0, ..., q_N) \in \mathbb{R}^{N+1}$.

- *Stochastic case:* Using expectation we define the stochastic functional

$$\mathcal{L}(X_t) = E\left[\int_a^b L(X_t, DX_t, t)\, dt\right]. \tag{6}$$

The study of such functionals is done by developing the associated calculus of variations. In the following section, we detail the stochastic calculus of variations and some applications. The main point is that critical points of the embedded functional are in general characterized by a differential equation in D over V. This equation is called the (V, D)-Euler-Lagrange equation. As an example, we have:

- *Discrete:* The discrete Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial x}(q_j, (\nabla_+ Q)_j) - \nabla_-\left(\frac{\partial L}{\partial v}(Q, (\nabla_+ Q))\right)_j = 0,$$

  for all $j = 1, ..., N - 1$, where

$$[\Delta_-(Q)]_i = \frac{Q_i - Q_{i-1}}{h},$$

  with $h = 1/N$, $i = 1, ..., N - 1$ and $[\Delta_+(Q)]_0 = 0$.

- *Stochastic:* The stochastic Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial x}(X_t, DX_t) - D_*\left[\frac{\partial L}{\partial v}(X_t, DX_t)\right] = 0, \tag{7}$$

where

$$D_* X_t := \lim_{h \to 0^+} E\left[\frac{X_t - X_{t-h}}{h} \mid \mathcal{F}_t \right].$$

As a consequence, we have for a given Euler-Lagrange equation two ways to extend it over V.

The first way is to make a direct embedding of the differential equation. For example, we have

- *Discrete:* The discrete embedding of the Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial x}(q_j, (\nabla_+ Q)_j) - \nabla_+ \left(\frac{\partial L}{\partial v}(Q, (\nabla_+ Q)) \right)_j = 0,$$

  for all $j = 0, ..., N-1$.

- *Stochastic:* The stochastic embedding of the Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial x}(X_t, DX_t) - D\left[\frac{\partial L}{\partial v}(X_t, DX_t) \right] = 0.$$

The second way is to embed the Lagrangian functional associated to the equation and to consider the embedded Euler-Lagrange equation.

These two possibilities can be resume by the following diagram:

$$\mathcal{L}(q) \xrightarrow{(V,D)-embedding} \begin{array}{c} \mathcal{L}_{emb}(Q) \\ \downarrow \\ (V,D)-EL \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{L}(q) \\ \downarrow \\ EL \end{array} \xrightarrow{(V,D)-embedding} Embedded\ EL$$

In general, the embedded Euler-Lagrange equation does not coincide with the (V, D)-Euler-Lagrange equation as can be seen from the discrete and stochastic embedding cases. This problem of non-unicity leads us to the following definition:

Definition 1 (Coherence): *A* (V, D)-embedding scheme is coherent if the following diagram commutes

$$\begin{array}{ccc} \mathcal{L}(q) & \xrightarrow{(V,D)-embedding} & \mathcal{L}_{(V,D)}(Q) \\ \downarrow & & \downarrow \\ EL & \xrightarrow{(V,D)-embedding} & Embedded - EL = (V,D) - EL \end{array}$$

The default of coherence in a given embedding scheme can have many origins. Most of them are related to the properties of the operator D used on V. We refer to ([6], [7]) for a discussion of this problem. An example of a coherent embedding is given by the non-differentiable embedding defined in [3].

4. Embedding and Symmetries

The notion of *symmetry* plays an important role both in physics and mathematics. Symmetries are defined as transformations of a certain system, which result in the same object after the transformation is carried out. They are mathematically described by one parameter groups of transformations. Their importance range from fundamental and theoretical aspects to concrete applications.

Constants of motion are another fundamental notion of physics and mathematics. Typically, they are used in the calculus of variations and optimal control to reduce the number of degrees of freedom, thus reducing the problems to a lower dimension.

Emmy Noether was the first to prove, in 1918, that these two notions are connected: when a system exhibits a symmetry, then a constant of motion exists. The celebrated Noether's theorem provide an explicit formula for such constants of motion.

For a given embedding (V, D) we can ask for the following problems:

- *Persistence of symmetries for the embedded system:* Formally, if $\Phi = \{\phi_s\}_{s \in \mathbb{R}}$ is a one parameter group of dieomorphisms such that L is invariant under Φ, i.e.

$$L(q, \dot{q}, t) = L\left(\phi_s(q), \frac{d}{dt}\phi_s(q), t\right),$$

do we have invariance of the embedded Lagrangian under Φ, i.e.

$$L(Q, DQ, t) = L(\phi_s(Q), D(\phi_s(Q))), t)\ ?$$

- (V, D)-*Noether theorem:* Assuming that the embedded Lagrangian $L(Q, DQ, t)$ is invariant under a one parameter group of diffeomorphisms, do we obtain an analogue of Noether's theorem ?

It is difficult to characterize the symmetries which are preserved under a given embedding, and we have only partial results in this direction. For example, if the group generated by the symmetry is linear, i.e. of the form $\phi_s : q \mapsto A_s \cdot q$, where A_s is a matrix, then the symmetry is preserved by the stochastic embedding. We refer to [6] for more details.

Assuming the existence of a symmetry group for the embedded functional, we obtain analogue of Noether's theorem. As an example, the stochastic Noether's theorem is as follow:

Theorem 1 (Stochastic Noether's theorem): Let F be the functional associated to L over $\Xi = \left\{X \in \Lambda^1, E\left[\int_0^T \left| L(X_t, \mathcal{D}_\mu X_t \right| dt\right] < \infty\right\}$. We assume that F is invariant under a one parameter group of dieomorphisms $\Phi = (\phi_s)_{s \in \mathbb{R}}$. Let $X^0 \in \Xi \cap \Lambda$ be a Λ^1-critical point of F. We denote by $Y_t(s) = (\Phi_s X^0)_t$. Then, we have

$$\frac{d}{dt} E\left[\partial_v L(X^0, \mathcal{D}X_t^0) \cdot \frac{\partial Y_t}{\partial s}(0)\right] = 0. \tag{8}$$

Such statement allows us to give the most natural analogue of a first integral for the embedded equation. Indeed, it is not in general easy to determine what can be called a conservation law or a first integral in the embedded case (see for example [7], [3], [6]). Having an analogue of the Noether's theorem in the embedded case lead to a definition. As an example, in the stochastic case a first integral is defined by:

Definition 2: Let X be a critical point of the stochastic Euler-Lagrange equation. A map $I : \Lambda^1 \to C^0(\mathbb{R}, \mathbb{R})$ is a first integral for X if $\frac{d}{dt} I(X_t) = 0$, i.e. $t \mapsto I(X_t)$ is constant.

We refer to [6] for more details.

5. EXAMPLE: STOCHASTIC EMBEDDING AND PARTIAL DIFFERENTIAL EQUATIONS

In this section, we define the stochastic analogue of characteristics for PDEs. We refer to [?] for more details.

5.1 Stochastic Calculus of Variations

We First Define the Stochastic Analogue of the Classical Functional.

Definition 3: Let L be an admissible Lagrangian function. Set

$$\Xi = \left\{X \in \Lambda^1, E\left[\int_0^T \left| L(X_t, \mathcal{D}_\mu X_t \right| dt\right] < \infty\right\}.$$

The functional associated to L is defined by

$$F:\left\{\begin{array}{lll}\Xi & \to & \mathbb{R} \\ X & \mapsto & E\left[\int_0^T L(X_t, DX_t)\,dt\right]\end{array}\right\}. \tag{9}$$

In what follows, we need a special notion which we will call L-adaptation:

Definition 4: Let L be an admissible Lagrangian function. A process $X \in \Lambda^1$ is said to be L-adapted if:

(i) $X \in \Xi$;

(ii) For all $t \in I$, $\partial_x L(X_t, \mathcal{D}_\mu X_t) \in L^2(\Omega)$;

(iii) $\partial_x L(X_t, \mathcal{D}_\mu X_t) \in \Lambda^1$.

The set of all L-adapted processes will be denoted by $\mathcal{A}_L$.

We introduce the following terminology:

Definition 5: Let Γ be a subspace of Λ^1 and $X \in \Lambda^1$. A Γ-variation of X is a stochastic process of the form $X + Z$, where $Z \in \Gamma$. Moreover set

$$\Gamma_\Xi = \{Z \in \Gamma, \forall X \in \Xi, Z + X \in \Xi\}.$$

We now define a notion of differentiable functional. Let Γ be a subspace of Λ^1.

Definition 3: Let L be an admissible Lagrangian function and F the associated functional. The functional F is called Γ-differentiable at $X \in \mathcal{A}_L$ if for all $Z \in \Gamma_\Xi$

$$F(X+Z) - F(X) = dF(X, Z) + R(X, Z), \tag{10}$$

where $dF(X, Z)$ is a linear functional of $Z \in \Gamma_\Xi$ and $R(X, Z) = o(\|Z\|)$.

A Γ-critical process for the functional F is a stochastic process $X \in \Xi \cap \mathcal{A}_L$ such that $dF(X, Z) = 0$ for all $Z \in \Gamma_\Xi$ such that $Z(a) = Z(b) = 0$.

5.2 Coherent Stochastic Embedding

As we already pointed out in the previous section, the stochastic embedding is not coherent if one use the full stochastic of variations, i.e. with $P = \Lambda^1$. In order to obtain a coherent embedding, we must restrict the set of variations. Let us introduce the space of Nelson differentiable processes:

$$\mathcal{N}^1 = \{X \in \Lambda^1, DX = D_* X\}. \tag{11}$$

Using $\mathcal{N}^1$-variations we have been able to prove the following result [6]:

Proposition 1: Let L be an admissible lagrangian with all second derivatives bounded. A solution of the equation

$$\frac{\partial L}{\partial x}(X_t, DX_t) - D\left[\frac{\partial L}{\partial v}(X_t, DX_t)\right] = 0, \tag{12}$$

called the Stochastic Euler-Lagrange Equation (SEL), is a $\mathcal{N}^1$-critical process for the functional F associated to L.

We have not been able to prove the converse of this proposition for $\mathcal{N}^1$-variations.

5.3 Stochastic Method of Characteristics

The classical method of characteristics for a PDE is to look for curves $s \mapsto (x(s), t(s))$ where $x(s)$ and $t(s)$ are solutions of an ordinary differential equation such that solutions $u(x, t)$ of the PDE satisfies

$$\frac{d}{ds}(u(x(s), t(s))) = F(x(s), t(s)),$$

where F is the non homogeneous part of the PDE.

In many cases, we can choose

$$\frac{dt}{ds} = 1$$

so that one is reduced to find a curve $t \mapsto x(t)$ satisfying the following ordinary differential equation

$$\frac{d}{ds}(u(x(t), t)) = F(x(t), t).$$

The method of characteristics does not work for parabolic PDEs and PDEs of mixed type like hyperbolic/parabolic (as for example the transport equation with diffusion).

Using the Nelson's derivatives D or D_* one can generalize this method. We say that a stochastic process X_t is a forward (resp. backward) characteristic for a given PDE if the stochastic process $u(X_t, t)$ satisfies

$$D(u(X_t, t))) = F(X_t, t) \qquad (\text{resp. } D_*(u(X_t, t))) = F(X_t, t)).$$

In the following section, we characterize the stochastic characteristics of the Navier-Stokes equation.

5.4 Stochastic Characteristics of the Navier-Stokes Equation

By the Navier-Stokes *equation* we mean the following equation:

$$\partial_t u + (u \cdot r)u = \nu\Delta u - \nabla p, \tag{13}$$

where $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the velocity field and $p : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is the pressure.

We first introduce the following class of diusion processes : The class Λ_c^2 composed of all the diusions $X^{(u,\sigma)}$ such that σ is constant, the drift u is C^2, bounded with all its second derivatives bounded, and $\nabla \log \rho_t$ wher ρ_t is the probability density has bounded second order derivatives. We can prove that $\Lambda_c^2 \subset \Lambda^2$ thanks to Prop. 2 p. 394 in [8]. Recall that a (u, σ)-diffusion with a constant diffusion coefficient σ is of the form:

$$X_t = X_0 + \int_0^t u(s, X_s)\, ds + \sigma W_t. \tag{14}$$

We recall that for $X^{b,\sigma} \in \Lambda^1$, there exists a measurable function b_* such that $D_* X_t = b_*(t, X_t)$. We call it the left velocity field of X.

The main result of ([5]) can be stated as follow:

Let $F([0, T] \times \mathbb{R}^d)$ be the space of measurable functions defined on $[0, T] \times \mathbb{R}^d$ and let $F([0, T] \times \Omega)$ be the space of measurable stochastic processes defined on $[0, T] \times \Omega$. Let us define the involution ϕ: $F([0, T] \times \mathbb{R}^d) \to F([0, T] \times \mathbb{R}^d)$ such that for all $t \in [0, T]$ and $x \in \mathbb{R}^d$, $(\phi u)(t, x) = -u(T - t; x)$. We denote by $\underline{u} := \phi u$.

We also define the time-reversal involution on stochastic processes: $r : F([0, T] \times \Omega) \to F([0, T] \times \Omega)$, $r(X)_t(\omega) = X_{T-t}(\omega)$. We denote by $\underline{X} := r(X)$.

Theorem 2: Let ν > 0, the backward stochastic characteristics of the Navier-Stokes equation over the class Λ_c^2 of diffusion processes $X^{(v,\sqrt{2\nu})}$ such that $v_* = u$ satisfies the Navier-Stokes equation, correspond to $\mathcal{N}^1$-critical point of the stochastic functional

$$X \mapsto E\left[\int_0^T L(X_t, DX_t)\, dt\right], \tag{15}$$

where L is the natural lagrangian $L(x, v) = \frac{v^2}{2} - p$.

Proof: As $X_t \in X^{(v,\sqrt{2\nu})}$ is a stochastic characteristic of the Navier-Stokes equation, we have

$$D_*(u(t, X_t) = -\nabla \underline{p}(t, X_t).$$

But $D_* X_t = v_* = u(t, X_t)$ where u satisfies the Navier-Stokes equation, so that the characteristic equation reduced to

$$D_*(D_* X_t) = -\nabla \underline{p}(t, X_t), \tag{16}$$

which concludes the proof.

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